# Forgetful Updating: Ignoring Small Signals and Approximating Big Ones

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**Abstract**

I study implications of imperfect information processing to optimal learning. The decision-maker receives informative signals each period and with some probability is asked to make a terminal action based on the signals received so far. The decision-maker is restricted to use a finite automaton no larger than a given size to process the signals, while randomization is allowed. In contrast to the previous literature that focuses on very low probability of termination and uses long-run average as the benchmark, I consider an information structure, a model of breakthroughs, in which analytical solutions are available for all probability values of termination. Results from that model are useful for more general setups and reveal two robust predictions regarding constrained optimal behaviour. First, it is optimal to ignore small (in terms of informativeness) signals. Second, big signals with similar strengths should be treated similarly but otherwise they are processed according to their approximate relative strengths.

**Key words:** Imperfect recall, bounded rationality, bounded memory, absent-minded, behavioural biases.

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# 1 Introduction

In recent years there has been a surge of interests in behavioural biases in economic agents' decision-making processes. One approach to understand such biases is to introduce frictions/capacity constraints in the agent's ability to process information, and derive behavioural biases as the optimal responses to those constraints. A case in point is the framework proposed by Hellman and Cover (1970) and its recent incarnation, Wilson (2014). These papers derive constrained optimal decision rules under limited memory modelled by finite automata.[1] Wilson (2014) shows that a rational decision-maker (DM) only responds to extreme signals but ignores any others, and cancels a positive extreme signal with a negative one (albeit sometimes with randomized schemes), regardless of their relative strengths. However, these impressive results are obtained only at the limit case where the expected time horizon for a decision is infinitely far away relative to the memory capacity, and it is not obvious whether and which of these results are robust when the expected time horizon is bounded away from that limit.

In this paper, I revisit these results and recover robust predictions from constrained optimal rules. I adopt the same framework as Wilson (2014), which features two possible states of nature, $H$ and $L$. In each period the DM receives an informative signal, and there is a chance that she has to make a terminal decision. Without memory constraints, the optimal decision rule is to take the optimal action with respect to the posterior from the signals received so far according to the Bayes' rule. Against this background environment, a memory constraint is imposed so that only decision rules that are implementable by a finite automaton of a given size is allowed. This consists of finitely many memory states and a transition rule that governs its evolution by specifying the next state to go to (which can be randomized) conditional upon the signal received. Each memory state is associated with an action, which would be taken if a terminal action is called for when the DM happens to land on that memory state.

Wilson (2014) focuses on the limit case where the probability of terminal action, $\eta$,

[1] There is a long history of using finite automata to model economic agents. Earlier works include Rubinstein (1986) on repeated games. More recently, Monte and Said (2014) extends the analysis of Hellman and Cover (1970) to changing worlds where state of nature may switch over time.

vanishes. Assuming that the distribution of signals has a full support, she proves two salient properties of the optimal finite automaton of a given size for $\eta$ sufficiently small. First, small signals are ignored in the sense that they do not trigger a transition to a different memory state. Surprisingly, however, all signals but the most informative signal in either direction are *small* signals, and hence the optimal automaton only reacts to extreme signals. Second, the memory states can be ranked in the sense that the extreme signal (the *high* signal) that increases the posterior on $H$ would cause a transition to a higher memory state, and the extreme signal that decreases it (the *low* signal) would cause a downward transition. Here comes the second surprise: any transition can only occur between adjacent memory states, and hence the effect of a high signal is similar to the effect of a low signal, even though they can have very different relative strengths in terms of Bayesian updating.

In contrast to Wilson (2014), I consider arbitrary termination probability. First I show that the decision problem in Wilson (2014) is intrinsically infinite: no finite automaton can implement the unconstrained optimal decision rule under the full-support condition. This infinite nature makes the model intractable and only asymptotic results are available as $\eta$ vanishes. In contrast, if one of the signals fully reveals one of the states of nature, then the unconstrained optimum is implementable with a finite automaton, whose size depends on the prior belief. I call such information structure a "model of breakthroughs," with the revealing signal being the "breakthrough" signal.[2]

In the model of breakthroughs I obtain analytical solutions for all $\eta$'s. These solutions reveal both robust predictions and rather fragile ones among Wilson (2014)'s results. For the robust prediction, I give a general sufficient condition for ignoring small signals, defined as those that are sufficiently uninformative, to be optimal, which is satisfied in the breakthrough model with arbitrary termination probability. This result is consistent with Wilson (2014)'s result for $\eta$ close to zero but here it is generalized to all $\eta$. However, two other features are not consistent with Wilson (2014)'s result. First, the memory states may not move to adjacent states but can jump to states far away. Second, the DM treats signals of similar strengths similarly, albeit in a lexicographical order. Specifically, when randomization is optimal, the stronger signal is more fluid in the sense that it moves the memory state to the next until the weaker one is completely

[2]This terminology is borrowed from Che and Mierendorff (2019).

sticky. However, both signals can be fluid even for $\eta$ arbitrarily small.

These results also hold for information structures where one signal is very informative but is not fully revealing. Specifically, as long as one signal is sufficiently informative, the optimal SFSA features jumps in memory states and "big" signals are treated similarly to extreme ones. The feature that small signals are ignored continue to hold. Note that these results are obtained under the full-support condition. To reconcile with Wilson (2014)'s findings, note that in her case the informative structure is fixed when $\eta$ is taken to zero, but in my case $\eta$ is fixed. These results then show that certain predictions are more robust, namely, ignoring small signals and treating signals with similar strengths similarly, while others do not hold for $\eta$ away from zero.

Finally, the paper also makes a methodological contribution by employing a new proof technique. In both Wilson (2014) and the paper she builds upon, Hellman and Cover (1970), the main results are obtained by considering the limit case where the payoff is effectively the long-run average and hence the use of the invariant distribution of the Markov chain derived from the transition of the optimal finite automaton is essential. In contrast, my results are obtained from the analytical solution of the constrained optimal rule in the model of breakthroughs, and then use the multi-self consistency to recover constrained optimal rules when signals are not fully revealing but very informative.

## 2 The Model

The model is essentially Wilson (2014). There are two states of nature, $\theta \in \{H, L\}$, and the prior probability over $\theta = H$ is $\mathbf{P}_0(H) = p_0$. The model has an unbounded number of period, and in each period, with probability $\eta$ the DM has to choose between two actions, $a \in A = \{a^H, a^L\}$, with utility function $u(a, \theta)$ given by

$$u(a^H, H) = u^H > 0,\ u(a^L, L) = u^L > 0,\ u(a^H, L) = 0 = u(a^L, H).$$

Once the action is chosen the game is over. With probability $1-\eta$ the game continues. Note that the chance to choose an action is exogenously given. The state of nature, however, is not observable to the DM. Instead, the DM can observe a sequence of signals, and the sequence is i.i.d. conditional on the state of nature. The set of possible realizations in each period are drawn from the set $\mathcal{S} = \{1, ..., S\}$, which are i.i.d. conditional on $\theta$ across periods. Conditional on $\theta$, $s \in \mathcal{S}$ occurs with probability $\mu_s^\theta$.

A decision rule is a function $D : \mathcal{S}^* \to A$, where $\mathcal{S}^*$ is the set of all partial histories of signal realizations, including the empty one. The unconstrained optimal rule can be fully characterize by the posterior, $p$: the optimal decision is to take $a^H$ whenever $p > p^* \equiv u^L/(u^H + u^L)$ and to take $a^L$ whenever $p < p^*$. The posterior is computed according to the Bayes rule, for which it is convenient to work with likelihood ratios. Define

$$\rho_0 = \frac{p_0}{1-p_0}, \ \rho^* = \frac{p^*}{1-p^*} = \frac{u^L}{u^H}, \ \text{and } \xi(s) = \frac{\mu_s^H}{\mu_s^L} \text{ for all } s = 1, ..S.$$

Normalize the labels so that $\xi(s)$ is increasing in $s \in \mathcal{S}$. I use $\rho$ to denote the likelihood ratio $p/(1-p)$ for a generic posterior $p$ on $H$, and from $\rho$ and signal $s$, the likelihood ratio for the new posterior is $\rho' = \rho\xi(s)$.

## Finite automata and implementation

Given the set of signals, $\mathcal{S}$, a stochastic finite-state automaton (SFSA) consists of a list $M = \langle Q, \tau, d, g \rangle$, where $Q$ is the set of *memory states*, $\tau : Q \times \mathcal{S} \to \Delta(Q)$ is the *transition rule*, $d : Q \to \Delta(A)$ is the *action rule*, and $g \in \Delta(Q)$ is the distribution over initial states. I use $\tau(q, s; q')$ to denote the transition probability from $q$ to $q'$ when receiving $s$, and $d(a, q)$ to denote the probability of taking action $a$ at memory state $q$ when called upon for terminal action. When the finite automaton is deterministic (abbreviated as DFSA), I use $\tau(q, x) = q'$ as the transition rule and $d(q) = a$ as the decision rule.

I first consider a two-signal world where $\mathcal{S} = \{h, \ell\}$ with $\xi(h) > 1 > \xi(\ell)$. The following result fully characterizes the signal structures for which a finite automaton can implement the unconstrained optimum.

**Proposition 2.1.** *Suppose that $\xi(h) < \infty$ and that $\eta \in (0, 1)$. Then, the unconstrained optimum is implementable with a finite automaton if and only if $\xi(\ell) = 0$.*

Proposition 2.1 is proved by the use of the Myhill-Nerode Theorem (Nerode, 1958), which gives full characterization of what finite automata of a given size can do in terms of partitions of the partial histories. For self-containment I give the relevant version of the theorem in the Appendix. Proposition 2.1 then shows that except for the case where one of the two signals fully reveals the state of nature, no finite automaton can implement the unconstrained optimum, and hence the problem is infinite by nature. The impossibility result can be easily extended to more than two signals, as more signals can only make the problem more complicated. As a benchmark, in the next section I study a model where one signal does fully reveal the state of nature, and the results from there are then used to obtain robust predictions in the more general setup.

# 3 A model of breakthroughs

Here I consider a two-signal structure, $\mathcal{S} = \{h, \ell\}$, with a revealing signal, $\mu_h^H = 1$, that is, $\xi(\ell) = 0$.[3] Later I will extend the analysis to allow for $\mu_h^H < 1$, and for more than two signals. To simplify notation, denote $\mu_\ell^L \equiv \mu$ with $\mu \in (0, 1)$. A low signal, $\ell$, fully reveals the state of nature $L$ (hence a *breakthrough*), while a high signal, $h$, only gradually increases the posterior on $H$. Without memory constraint, the optimal rule would dictate that action $a^L$ to be taken whenever a low signal appears and thereafter, while $a^H$ is taken only after sufficiently many high signals without seeing any low. As shown later, this can be implemented by a finite automaton with sufficiently many memory states. I will also characterize the constrained optimal rule when memory is constrained.

First, suppose that $\rho_0 \geq \rho^*$. In this case, the unconstrained optimum can be implemented by a two-state DFSA with $Q = \{q_H, q_L\}$, $\tau(q_H, h) = q_H$, $\tau(q_H, \ell) = q_L$, $\tau(q_L, h) = q_L = \tau(q_L, \ell)$, and $d(q_H) = a^H$ and $d(q_L) = a^L$, as depicted in Figure 1. Thus, the memory state $q_L$ is self-absorbing for both signals, and the action is $a^L$; $q_H$ is absorbing only for signal-$h$, and the action is $a^H$. This DFSA, labelled $M_2^b$, with $q_H$ as the initial state, implements the unconstrained optimum: since $p_0 \geq p^*$, it is optimal to take $a^H$ when receiving high signals, and it is optimal to switch to $a^L$ and continue

[3]By assuming that $\mu_s^\theta > 0$ for all $s$ and for both $\theta = H, L$, Wilson (2014)'s analysis excludes this case. However, as we will see below, this is not a trivial case and it helps us solve some of the cases under the full support condition, but gives a rather different implications than what Proposition 4 in Wilson (2014) suggests.

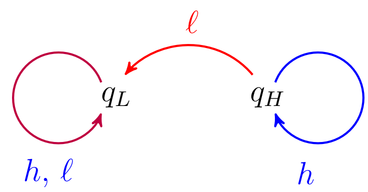


Figure 1: The DFSA, $M_2^b$, that implements unconstrained optimum when $p_0 \geq p^*$

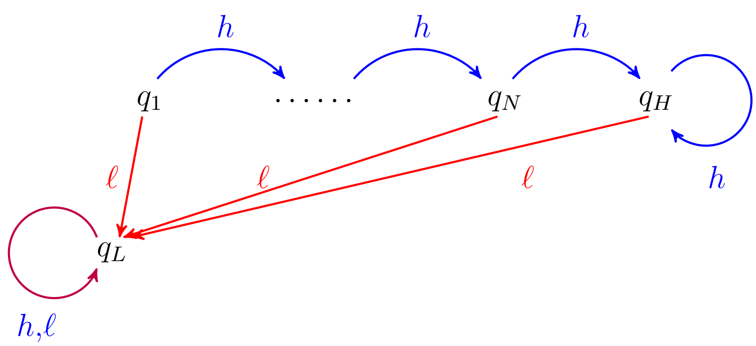


Figure 2: The DFSA, $M_{N+2}^b$, that implements unconstrained optimum when $p_0 < p^*$

using that whenever a low signal is received. Note that this implementation does not depend on $\eta$.

Suppose that now $\rho_0 < \rho^*$. In this case, the unconstrained optimum can still be implemented by a DFSA, but its size depends on the distance between $\ln \rho_0$ and $\ln \rho^*$ relative to $\ln \xi(h)$. In particular, let

$$N = N(\rho_0) \equiv \left\lceil \frac{\ln \rho^* - \ln \rho_0}{\ln \xi(h)} \right\rceil. \tag{1}$$

Then, the optimal DFSA requires $N + 2$ states, given by $Q = \{q_H, q_L, q_1, ..., q_N\}$, $\tau(q_L, \ell) = q_L = \tau(q_L, h)$, $\tau(q_i, h) = q_{i+1}$ and $\tau(q_i, \ell) = q_L$ for $i = 1, ..., N-1$, $\tau(q_N, h) = q_H$ and $\tau(q_N, \ell) = q_L$, and $\tau(q_H, h) = q_H$ and $\tau(q_H, \ell) = q_L$, as depicted in Figure 2, with $q_1$ as the initial memory state. The action rule is given by $d(q_L) = d(q_i) = a^L$ for all $i = 1, ..., N$, and $d(q_H) = a^H$. This DFSA, labelled $M_{N+2}^b$, implements the unconstrained optimum for all $\eta \in (0, 1)$. When $N = 0$, this is equivalent to $M_2^b$, and hence we can set $N = 0$ in (1) for $\rho_0 \geq \rho^*$.

These results then show that whether the memory constraint, $|Q| \leq K$, binds or not depends only on $\rho_0$, and it binds if and only if $K < N(\rho_0) + 2$. Conversely, for a given $K$, let $\rho_0^K$ be the smallest $\rho_0$ such that $N(\rho_0) + 2 \leq K$. Then, the memory constraint is binding if and only if $\rho_0 < \rho_0^K$. The following proposition shows that strict randomization is optimal if and only if the memory constraint is binding and the optimal SFSA is not trivial.

**Proposition 3.1.** *Suppose that $\mathcal{S} = \{h, \ell\}$ with $\xi(\ell) = 0$ and $\xi(h) \in (0, 1)$. Let $K \geq 2$ and $\eta \in (0, 1)$ be given. There exists $\underline{\rho}_0 < \rho_0^K$ such that for all $\rho_0 \in (\underline{\rho}_0, \rho_0^K)$, the optimal SFSA uses strict randomization. For smaller $\rho_0$, optimal SFSA can be deterministic, but it must be the trivial one that dedicates $a^L$ all the time. Moreover, $\underline{\rho}_0$ converges to zero as $\eta$ goes to zero.*

Proposition 3.1 is proved by considering the following SFSA, deviating from $M_K^b$ by having $\tau(q_i, h; q_i) = \alpha = 1 - \tau(q_i, h; q_{i+1})$ for $i = 1, ..., K$, where $q_{K+1} = q_H$, while all the other transitions follow the same rule as $M_K^b$. This SFSA is denoted by $M_K^b(\alpha)$ and $M_K^b = M_K^b(0)$. For any $\rho_0 < \rho_0^K$, the optimal DFSA is either $M_b^K$ or the one that takes $a^L$ all the time. The proof shows that it is optimal to set $\alpha > 0$ and hence randomization is optimal relative to $M_K^b$. Thus, to show that optimal SFSA uses strict randomization, one only needs to beat the DFSA that takes $a^L$ all the time, whose payoff is $(1 - p_0)u^L$. Thus, to prove the existence of $\underline{\rho}_0 < \rho_0^K$, I show that the payoff from $M_K^b(\alpha)$ is strictly higher than $(1 - p_0)u^L$ for $\rho_0$ close to $\rho_0^K$ but smaller. In fact, one can show that the optimal SFSA indeed takes the form of $M_K^b(\alpha)$ for a range of priors below $\rho_0^K$, and a sketch of proof for this result can be found in the proof section. Finally, as $\eta$ converges to zero, the payoff from $M_K^b(\alpha)$ for any $\alpha < 1$ converges to $p_0 u^H + (1 - p_0)u^L$, the highest payoff possible.

# 4 Robust predictions

In this section I extend Proposition 3.1 to two settings. First, I consider environments with more than two signals, and show that small signals are ignored but big signals are treated similarly under the constrained optimal rule. Second, I consider the case where the low signal does not fully reveal the state of nature but is very informative. These results allow for a direct comparison against conclusions in Wilson (2014) and reveal the robust ones.

I first need some structural results that extend those in Wilson (2014) to characterize locally optimal rule under the memory constraint. Given a state of nature $\theta$ and a memory state $q \in Q$, the expected payoff accumulated from $q$ conditional on $\theta$ is then

$$\begin{aligned}
&\eta g(q)\left[\sum_{a\in A} d(a;q)u(a,\theta)\right] + \eta(1-\eta)\sum_{q_1,s_1\in\mathcal{S}} g(q_1)\tau(q_1,s_1;q)\mu^\theta_{s_1}\left[\sum_a d(a;q)u(a,\theta)\right]\\
&+\eta(1-\eta)^2\sum_{q_1,q_2\in Q,s_1,s_2\in\mathcal{S}} g(q_1)\tau(q_1,s_1;q_2)\mu^\theta_{s_1}\tau(q_2,s_2;q)\mu^\theta_{s_2}\left[\sum_a d(a;q)u(a,\theta)\right] + .....\\
&= f(q|\theta)\left[\sum_a d(a;q)u(a,\theta)\right],
\end{aligned} \tag{2}$$

where

$$f(q|\theta) = \sum_{T=1}^{\infty}\eta(1-\eta)^{T-1}\left[\sum_{(q_1,....q_{T-1}),(s_1,...,s_{T-1}),q_T=q} g(q_1)\prod_{t=1}^{T-1}\mu^\theta_{s_t}\tau(q_t,s_t;q_{t+1})\right]. \tag{3}$$

As noted in Wilson (2014), $f(q|\theta)$ is the stationary distribution under the transition probability from $q'$ to $q$ given by

$$T^\theta(q';q) = \sum_{s\in\mathcal{S}}[\eta g(q) + (1-\eta)\mu^\theta_s\tau(q',s;q)]. \tag{4}$$

Wilson (2014) then defines the "belief" at $q \in Q$ as

$$p(q) = \frac{p_0 f(q|H)}{p_0 f(q|H) + (1-p_0)f(q|L)} \text{ and } p(q,s) = \frac{p_0 f(q|H)\mu^H_s}{p_0 f(q|H)\mu^H_s + (1-p_0)f(q|L)\mu^L_s}, \tag{5}$$

with the associated likelihood ratios given by

$$\rho(q) = \rho_0 f(q|H)/f(q|L) \text{ and } \rho(q,s) = \rho(q)\xi(s).$$

To characterize an optimal SFSA, I use $V_q(\theta)$ to denote the continuation value at memory state $q$ conditional on the state of nature being $\theta$. Two memory states are called *equivalent* if they share the same transition rules to any other states or their equivalents, and have the same decision rule.

**Proposition 4.1.** *Suppose that $M$ is an optimal SFSA among those of size $|Q| = K$ without equivalent states, and we rank the memory states in $M$ according to*

$$\rho(q_1) \leq \rho(q_2) \leq \cdots \leq \rho(q_K),$$

*with the convention that if $\rho(q_i) = \rho(q_{i+1})$ then $V_{q_i}(H) \leq V_{q_{i+1}}(H)$. Let $\Delta V^\theta_{i,j} = V^\theta_{q_i} - V^\theta_{q_j}$.*

1. *$\Delta V^H_{i,j} < 0$ and $\Delta V^L_{i,j} > 0$ for all $i < j$, and $\Delta V^H_{j,i}/\Delta V^L_{i,j} \geq \Delta V^H_{k,j}/\Delta V^L_{j,k}$ for all $i < j < k$.*

2. *Define $\bar{\rho}_i = \Delta V^L_{i,i+1}/\Delta V^H_{i+1,i}$, $i = 1, ..., K-1$. Then, in $M$,*

   *(a) for each $q_i$,*
   $$\rho(q_i) \in [\bar{\rho}_{i-1}, \bar{\rho}_i]; \tag{6}$$

   *(b) $\tau(q, s; q_i) > 0$ only if*
   $$\rho(q, s) \in [\bar{\rho}_{i-1}, \bar{\rho}_i], \tag{7}$$
   *where $\bar{\rho}_0 = 0$ and $\bar{\rho}_K = \infty$;*

   *(c) $d(q_i, a^h) > 0$ only if $\rho(q_i) \geq \rho^*$ and $d(q_i, a^\ell) > 0$ only if $\rho(q_i) \leq \rho^*$.*

*Conversely, if a DFSA $M$ satisfies $\rho(q, s) \in (\bar{\rho}_{i-1}, \bar{\rho}_i)$ for all $q \in Q$ with $\tau(q, s) = q_i$, and $d(q) = a^H$ for all $q \in Q$ with $\rho(q) > \rho^*$ and $d(q) = q^L$ for all $q \in Q$ with $\rho(q) < \rho^*$, then $M$ is locally optimal.*

The first part of Proposition 4.1 follows directly from Corollary 1 and Lemma 1 in Wilson (2014), but the converse for DFSA to be locally optimal is new. Proposition 4.1 states that in the optimal SFSA, the DM essentially uses the beliefs $\rho(q)$ to decide the optimal transition and the optimal actions if called upon. Note that Proposition 4.1 applies to *all* information structures, including the model of breakthroughs and information structures that feature full supports. Now we are ready to extend the analysis from Section 3 to more general settings.

## More than two signals

First I consider environments with more than two signals. The first result is that *small* signals are ignored. A signal $s$ is *small* if $\xi(s)$ is close to one. Moreover precisely, consider the following perturbation. Begin with a signal structure $(\bar{\mu}_s^H, \bar{\mu}_s^L)_{s\in\mathcal{S}}$ in which $\xi(t) = 1$ for some fixed signal $t \in \mathcal{S}$. Then, consider a nearby information structure where $\max_{s\in\mathcal{S},\theta=H,L} |\mu_s^\theta - \bar{\mu}_s^\theta| < \epsilon$ and hence $\xi(t)$ is also close to one.

**Proposition 4.2.** *Fix some $s \in \mathcal{S}$. Suppose that in all optimal SFSA $M$ under the constraint $|Q| = K$ under $(\bar{\mu}_s^H, \bar{\mu}_s^L)_{s\in\mathcal{S}}$ with $\xi(t) = 1$ for some $t$, $\bar{\rho}_{i-1} < \rho(q_i) < \bar{\rho}_i$ for all $i$. Then, there exists some $\epsilon > 0$ such that for all information structure perturbing the original one with $\max_{s\in\mathcal{S},\theta=H,L} |\mu_s^\theta - \bar{\mu}_s^\theta| < \epsilon$, any optimal SFSA ignores $t$, i.e., $\tau(q, t; q) = 1$ for all $q \in Q$.*

Proposition 4.2 gives a sufficient condition for small signals to be ignored, which is satisfied in the model of breakthroughs and the symmetric information structure considered in Wilson (2014), Appendix E. For small $\eta$'s, this result is weaker than Wilson (2014), who shows that the optimal SFSA responds only to extreme signals. However, in Proposition 4.2 there is no requirement for $\eta$ to be small, and, as we will see later, in the model of breakthroughs the condition mainly depends on the prior.

Now let's turn to big signals, and here I confine my analysis to the model of breakthroughs. My methodology here is to introduce a third signal into the two-signal environment, with $\mathcal{S} = \{h, h', \ell\}$ and $\xi(\ell) = 0$ and $\xi(h) > \xi(h') > 1$ but the two are close.[4] The signal $h'$ is "big" in the sense that it is almost as informative as $h$. When $\xi(h) = \xi(h')$, any optimal SFSA simply treats $h$ exactly as $h'$, and when randomization is optimal, it splits the transition probability between the two arbitrarily.

Now consider $\xi(h') < \xi(h)$ but close. For a given $K$, first consider the case where $K > N + 2$ with $N$ given by (1). In this case, for all $\rho_0 \geq \rho_0^K$, there is a range $[\xi(h) - \epsilon, \xi(h)]$ for some $\epsilon > 0$ such that if $\xi(h')$ is in that range, the optimal SFSA is $M_{N+2}^b$, now interpreted as $\tau(q_i, h) = q_{i-1} = \tau(q_i, h')$ for all $i = 2, ..., N$ and $\tau(q_1, h) = q_H = \tau(q_1, h')$. For lower $\rho_0$'s, there is range in which the optimal SFSA takes the form

[4] Formally, begin with the benchmark case where $\mu_{h'}^H = 1 - \mu_h^H \in (0, 1)$, and $\mu_h^L = \mu_h^H(1 - \mu)$ and $\mu_{h'}^L = \mu_{h'}^H(1 - \mu)$, with $\mu_\ell^L = \mu$.

$M^b_K(\alpha)$, where the randomization at $q_i$ can be split arbitrarily between the two signals as long as

$$\mu^H_h \tau(q_i, h; q_{i-1}) + \mu^H_{h'} \tau(q_i, h'; q_{i-1}) = \alpha \tag{8}$$

for all $i = 1, ..., N$, with $q^o = q_H$. That is, the *average* probability of staying at $q_i$ is equal to $\alpha$ across the signals $h$ and $h'$. Let $M^b_K(\beta, \beta')$ denote the SFSA such that $\tau(q_i, h; q_{i-1}) = \beta$ and $\tau(q_i, h'; q_{i-1}) = \beta'$ for all $i = 1, ..., K-2$, with $q^o = q_H$. The following proposition shows that, whenever $\xi(h')$ close to $\xi(h)$, the optimal SFSA has the form $M^b_K(\beta, \beta')$, with the tie in (8) broken in a particular manner.

**Proposition 4.3.** *Let $\mathcal{S} = \{h, h', \ell\}$ with $\xi(\ell) = 0$ and let $K$ be the memory constraint.*

1. *For any $\rho_0 > \rho^K_0$, there exists $\epsilon > 0$ such that under the perturbed information structure with $\xi(h') \in [\xi(h) - \epsilon, \xi(h)]$, $M^b_K$ is the optimal SFSA.*

2. *Suppose that $\rho_0 < \rho^K_0$ and that the optimal SFSA has the form $M^b_K(\alpha)$ when $\xi(h') = \xi(h)$. There exists $\epsilon > 0$ such that under the perturbed information structure with $\xi(h') \in [\xi(h) - \epsilon, \xi(h))$, the optimal SFSA has the form $M^b_K(\beta, \beta')$ with optimal $\beta' = \tau(q_i, h'; q_{i+1}) > 0$ for all $i$. Moreover, in the optimal $M^b_K(\beta, \beta')$, $\beta > 0$ only if $\beta' = 1$.*

According to Proposition 4.3 (1), when $\xi(h')$ is close to $\xi(h)$ but slightly lower, the optimal SFSA treats both signals $h$ and $h'$ exactly the same when the optimal SFSA is deterministic. According to Proposition 4.3 (2), when randomization is optimal, it would first occur when receiving $h'$ before it occurs when receiving $h$; in other words, randomization between staying in the current memory state and the next when receiving $h$ can only happen when it is optimal to stay in the current memory state for sure when receiving $h'$. Note that Proposition 4.3 holds for any $\eta \in (0, 1)$, not just for low $\eta$'s.

## No extreme signals

Here I return to the two-signal world but consider the case where $\mu^H_h < 1$ but close, and hence the signal $\ell$ is no longer a breakthrough but still a strong signal for state of nature $L$.

**Proposition 4.4.** *Suppose that $\mathcal{S} = \{h, \ell\}$ and that $\mu_\ell^L \in (0,1)$. Let $K \geq 3$ be given.*

1. *Let $\eta \in (0,1)$ be given. For any $p_0 < p^*$ such that the optimal SFSA has the form $M_K^b(\alpha)$ with $\alpha \in [0,1)$, there exists $\epsilon > 0$ such that for all $\mu_h^H \in (1-\epsilon, 1]$, the optimal SFSA also has the form $M_K^b(\alpha)$.*

2. *For any given $\mu_h^H > 1 - \mu_\ell^L$ such that $\xi(h)^{K-1}\xi(\ell) < 1$,*

    (a) *if $\ln \rho^* - N \ln[\xi(h)] \leq \ln \rho_0 < \ln \rho^* - (N-1)\ln[\xi(h)]$ for some $N \leq K-2$, then $M_{N+2}^b$ is locally optimal for $\eta$ sufficiently large;*

    (b) *if $\ln \rho^* - M \ln[\xi(h)] < \ln \rho_0 < \ln \rho^* - (K-2)\ln[\xi(h)]$, where $M = \lceil \frac{-\ln[\xi(\ell)]}{\ln[\xi(h)]} \rceil$, then for some $\alpha \in (0,1)$, $M_K^b(\alpha)$ is locally optimal for $\eta$ sufficiently large.*

According to Proposition 4.4 (1), for any given $\eta \in (0,1)$, there is a range of information structures that satisfy the full-support, the optimal SFSA takes the form of $M_K^b(\alpha)$, and hence, when receiving signal $\ell$ it is optimal to transit all the way back to $q_L$ for all memory states, including the highest. This is in great contrast to the conclusion from Wilson (2014), who shows that transition only occurs between adjacent memory states for small $\eta$'s. The difference, of course, lies in the order according which the limit is taken. In Wilson's case it is to fix $\mu_h^H < 1$ and to take $\eta$ to zero, in my case it is to fix $\eta$ (no matter how small) and to consider $\mu_h^H$ close to one.

Now, one may wonder how close does $\mu_h^H$ need to be around one. Proposition 4.4 (2) shows that this depends on the prior, $p_0$. As mentioned above, for $\eta$ very close to zero, Wilson (2014) shows that $M_K^b(\alpha)$ cannot be optimal, even locally. However, Proposition 4.4 (2) shows that $M_K^b(\alpha)$ is locally optimal for $\eta$ relatively high for a range of priors, depending on how strong the signal $\ell$ is relative to signal $h$.

Finally, Proposition 4.4 can be easily extended to environments with more than two signals. In particular, Proposition 4.2 would hold under the assumptions in Proposition 4.4 and hence small signals will be ignored, as well as Proposition 4.3 and hence signals with similar strengths will be treated similarly.

# 5 Conclusion

An important point in Wilson (2014) is to establish that potential behavioural biases are in fact rational responses to decision-makers' imperfections in information processing abilities. Here I demonstrated that these implications do depend not only on the constraints imposed by such imperfections, but also the underlying environment; in particular, in the context of her model, these depend on the probability of terminal actions. My results suggest that two behaviour implications are robust: first, it is almost universally true that constrained optimal rule ignores small signals; second, the DM should treat big signals according to their informativeness, and, when it comes to optimal randomization, use a lexicographical order.

# Appendix: Proofs

Before the proof of Proposition 2.1, I first give a version of the Myhill-Nerode theorem adopted to the framework in the paper. To do so, I need to introduce some concepts first. A relation over the set of partial histories of signal realizations, $R \subset \mathcal{S}^* \times \mathcal{S}^*$, is called *right-invariant* if

$$\mathbf{x} R \mathbf{y} \;\longleftrightarrow\; (\mathbf{x} \circ \mathbf{z}) R (\mathbf{y} \circ \mathbf{z}) \text{ for all } \mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathcal{S}^*, \tag{9}$$

where $\circ$ denotes concatenation. Given a decision rule $D$, $\mathcal{L}_\theta^D$ denotes the set of partial histories under which action $a^\theta$ is taken, $\theta = H, L$.

**Theorem 5.1** (Myhill-Nerode Theorem)**.** *The rule $D$ can be implementable by a DFSA iff there is a right-invariant equivalence relation $R$ with finitely many equivalence classes such that $\mathcal{L}_\theta^D$ is a union of some of those equivalence classes for both $\theta = H, L$.*

The equivalence classes correspond to the memory states in the corresponding DFSA, and the equivalence classes that make up $\mathcal{L}_\theta^D$ consist of those action states where action $a^\theta$ is taken, for both $\theta = H, L$. Thus, the DFSA gives a finite partition of partial histories that captures the finiteness of the DM's memory capacity. The right-invariance condition captures the fact that if the DFSA enters the same memory state after two different partial histories, then it will end up in the same memory state (although not necessarily the same as the original one) after any consecutive partial history.

## Proof of Proposition 2.1

Note that the "if" part is proved in the analysis of the model of breakthroughs. For the "only if" part, under the assumption of two signals with $\xi(h) \in (1, \infty)$ and $\xi(\ell) \in (0, 1)$, I show that under the unconstrained optimal rule $D$, any right-invariant equivalence relation $R$ such that $\mathcal{L}_\theta^D$ is a union of some of those equivalence classes for both $\theta = H, L$, it must have infinitely many equivalence classes. The "only if" part then follows from the Myhill-Nerode theorem.

Now, let $R$ be such an equivalence relation. Without loss of generality I assume that $\rho_0 \le \rho^*$. Now, I construct an infinite sequence of partial histories, $\mathbf{x}^1, \mathbf{x}^2, ..., \mathbf{x}^n, ...$

and show that each has to be of a different equivalence class. The construction is simple: each $\mathbf{x}^n$ consists of $m_n$ $h$-signals, with the sequence $\{m_n\}_{n=1}^{\infty}$ determined as follows. Let $m_1$ be the smallest integer $m \geq 1$ such that

$$\ln(\rho_0) + m \ln[\xi(h)] > \ln(\rho^*).$$

Given $m_n$, $m_{n+1}$ is the smallest integer $m$ such that

$$(m - m_n) \ln[\xi(h)] > -\ln[\xi(\ell)]. \tag{10}$$

This implies that $m_n$ strictly increases with $n$. Now we show that, for each $n \geq 2$, $\mathbf{x}^n$ must belong to a different equivalence class other than the one $\mathbf{x}^i$ belongs to for all $i < n$. To see this, let $k$ be the smallest integer such that

$$\ln(\rho_0) + m_{n-1} \ln[\xi(h)] + k \ln[\xi(\ell)] < \ln(\rho^*).$$

This implies that $D(\mathbf{x}^i \circ \mathbf{y}) = a^L$ for all $i < n$, where $\mathbf{y}$ consists of $k$ $\ell$-signals. However, by (10),

$$\ln(\rho_0) + m_n \ln[\xi(h)] + k \ln[\xi(\ell)] > \ln(\rho_0) + m_{n-1} \ln[\xi(h)] + (k-1) \ln[\xi(\ell)] \geq \ln(\rho^*),$$

and hence $D(\mathbf{x}^n \circ \mathbf{y}) = a^H$. Thus, by right-invariance, $\mathbf{x}^n$ must belong to a different cell than $\mathbf{x}^i$ for all $i < n$.

## Proof of Proposition 3.1

First we show that if $K \leq N + 2$, then the optimal DFSA is either $M_K^b$, or to always take $a^L$ (which can be implemented with a DFSA with only one memory state, of course). So fix some $K \leq N + 2$. Now, consider the set of DFSA's which dedicates action $a^H$ for some memory state. We use $q_H$ to denote the memory state at which $d(q_H) = a^H$. Since $\ell$-signal is fully revealing, after seeing it the optimal thing to do is to take $a^L$ always. So it is optimal to have a memory state $q_L$ for which $d(q_L) = a^L$ and $\tau(q_L, h) = q_L$, and $\tau(q, \ell) = q_L$ for any $q \in Q$, that is, entering $q_L$ signals an $\ell$-signal has been received. Given this, we only need to have one memory state for action $a^H$, with $\tau(q_H, h) = q_H$. If we have two, it is optimal then to transit to one another when seeing $h$ only, and hence the two memory states are replicates of one another. Thus, the optimal DFSA takes the form of $M^b_{k+2}$. Now, we show that for all $k \leq N$, the expected payoff from $M^b_{k+2}$ increases with $k$.

Routine calculation shows that the value functions are given by $V_{q_L}(L) = u^L$, $V_{q_H}(H) = u^H$, and

$$\begin{aligned} V_{q_i}(H) &= (1-\eta)^{k-i+1}u^H; && (11)\\ V_{q_H}(L) &= \frac{(1-\eta)\mu u^L}{1-(1-\eta)(1-\mu)}; && (12)\\ V_{q_i}(L) &= \left\{1 - \frac{\eta(1-\eta)^{k-i+1}(1-\mu)^{k-i+1}}{1-(1-\eta)(1-\mu)}\right\}u^L. && (13)\end{aligned}$$

Now, the expected payoff from $M^b_{k+2}$ is better than that from $M^b_{k+1}$ if and only if

$$\begin{aligned} & p_0(1-\eta)^k u^H + (1-p_0)\left\{1 - \frac{\eta(1-\eta)^k(1-\mu)^k}{1-(1-\eta)(1-\mu)}\right\}u^L \\ \geq\ & p_0(1-\eta)^{k-1}u^H + (1-p_0)\left\{1 - \frac{\eta(1-\eta)^{k-1}(1-\mu)^{k-1}}{1-(1-\eta)(1-\mu)}\right\}u^L,\end{aligned}$$

which is equivalent to $\rho_0\xi(h)^{k-1} \leq \rho^*$.

Now I move to the case with randomization. At $\rho_0 = \rho_0^K$, we know that the optimal SFSA is $M^b_K$, but at that boundary point, for each $i = 1, ...., K-2$, $\rho(q_i, h) = \bar{\rho}_{i+1}$, a fact that can be verified by the formula (5) for $\rho(q_i, h)$, that is, the DM is indifferent between staying at $q_i$ and moving to $q_{i+1}$ when seeing an $h$-signal. By continuity and by Proposition 4.1, for $\rho_0$ slightly below $\rho_0^K$, the only relevant randomization has $\tau(q_i, h; q_i) > 0$ but $\tau(q_i, h; q_i) + \tau(q_i, h; q_{i+1}) = 1$, that is, randomizing between staying at $q_i$ and moving to $q_{i+1}$. Accordingly, the optimal SFSA would take the form $M^b_K(\alpha_1, ..., \alpha_{K-2})$, where $\tau(q_i, h, q_i) = \alpha_i = 1 - \tau(q_i, h, q_{i+1})$, $i = 1, ..., K-2$. First we compute the value functions under $M^b_K(\alpha_1, ..., \alpha_{K-2})$:

$$\begin{aligned} V_{q_i}(H) &= \left[\prod_{j=i}^{K-2}\frac{1-\alpha_j}{1-(1-\eta)\alpha_j}\right](1-\eta)^{K-i-1}u^H; && (14)\\ V_{q_H}(L) &= \frac{(1-\eta)\mu u^L}{1-(1-\eta)(1-\mu)}; && (15)\\ V_{q_i}(L) &= \left\{1 - \left[\prod_{j=i}^{K-2}\frac{1-\alpha_j}{1-(1-\mu)(1-\eta)\alpha_j}\right]\frac{\eta(1-\eta)^{K-i-1}(1-\mu)^{K-i-1}}{1-(1-\eta)(1-\mu)}\right\}u^L && (16)\end{aligned}$$

Note that this implies that the ex ante payoff, $p_0V_{q_1}(H) + (1-p_0)V_{q_1}(L)$, is symmetric in $(\alpha_1, ..., \alpha_{K-2})$ and supermodular, and hence it is optimal to set $\alpha_i = \alpha$ for all $i$. By doing so, the ex ante payoff is

$$\begin{aligned} F(\alpha) &= p_0 \left[\frac{1-\alpha}{1-(1-\eta)\alpha}\right]^{K-2} (1-\eta)^{K-2} u^H \qquad (17)\\ &+ (1-p_0)\left\{1 - \left[\frac{1-\alpha}{1-(1-\mu)(1-\eta)\alpha}\right]^{K-2} \frac{\eta(1-\eta)^{K-2}(1-\mu)^{K-2}}{1-(1-\eta)(1-\mu)}\right\} u^L. \end{aligned}$$

Thus,

$$\begin{aligned} F'(\alpha) &= p_0 \left[\frac{1-\alpha}{1-(1-\eta)\alpha}\right]^{K-3} \frac{-(K-2)\eta(1-\eta)^{K-2}u^H}{[1-(1-\eta)\alpha]^2} \qquad (18)\\ &+ (1-p_0)\left[\frac{1-\alpha}{1-(1-\mu)(1-\eta)\alpha}\right]^{K-3} \frac{(K-2)\eta(1-\eta)^{K-2}(1-\mu)^{K-2}u^L}{[1-(1-\eta)(1-\mu)\alpha]^2} \qquad (19) \end{aligned}$$

Thus, $F'(\alpha) > 0$ at $\alpha = 0$ if and only if

$$\frac{p_0}{1-p_0} < \frac{u^L}{u^H}(1-\mu)^{K-2},$$

that is, if and only if the memory constraint is binding. Since the best DFSA is $M_K^b = M_K^b(0)$ (other than taking $a^L$ all the time), this shows that strict randomization is better. Note that this also shows that $M_K^b(\alpha)$ is the optimal SFSA for a range of priors below $\rho_0^K$. Finally, I show the existence of $\underline{\rho}_0$ above which taking $a^L$ all the time is not optimal. For each $\rho_0 \leq \rho_0^K$, let $W(\rho_0)$ be the payoff from $M_K^b(\alpha)$ with the optimal $\alpha$. By the Theorem of Maximum, $W(\rho_0)$ is continuous in $\rho_0$. Now, at $\rho_0 = \rho_0^K$, since $M_K^b$ implements the unconstrained optimum, we have $W(\rho_0^K) > (1-p_0)u^L$, the latter being the payoff from taking $a^L$ always. Thus, by continuity, there exists $\underline{\rho}_0 < \rho_0^K$ such that $W(\rho_0) > (1-p_0)u^L$ for all $\rho_0 \in (\underline{\rho}_0, \rho_0^K]$.

Finally, from (17) it is straightforward to verify that

$$\lim_{\eta \to 0} F(\alpha) = p_0 u^H + (1-p_0)u^L$$

for any $\alpha \in [0, 1)$.

## Proof of Proposition 4.1

As mentioned, the first part follows from Wilson (2014), but I note a crucial observation. For a given SFSA, one can write the ex ante payoff $V$ as a function of the transition probabilities $\tau(q,s,q')$, $q,q,\in Q$ and $s\in\mathcal{S}$. For a given $(q,s,q')$ and denote $\pi=\tau(q,s,q')$, then

$$\frac{\partial}{\partial\pi}V=\left[p_0 f(q|H)\mu_s^H+(1-p_0)f(q|L)\mu_s^L\right]\{p(q,s)V_{q'}(H)+[1-p(q,s)]V_{q'}(L)\}\,. \quad (20)$$

Now I show the converse. Suppose that $M$ is a DFSA that satisfies the conditions. I show that there exists $\epsilon>0$ such that for all SFSA $M'$ such that $|\tau(q,s;q')-\tau'(q,s;q')|\leq\epsilon$ for all $q,q'\in Q$ and $s\in\mathcal{S}$, $M$ has a higher expected payoff than $M'$. Since the beliefs $p(q,s)$ are continuous in the transition probabilities, choose $\epsilon$ small so that the inequalities are preserved strictly.

Now, consider choice variables $\pi_{q'}=\tau(q,s;q')$ for each $q'\in Q$. Suppose that in the DFSA $M$ we have $\rho(q,s)\in(\bar\rho_{i-1},\bar\rho_i)$ and hence $\pi_{q_i}=1$, that is, $\tau(q,s)=q_i$. We impose the constraint $\pi_{q'}\in[0,\epsilon]$ and $\sum_{q'\in Q}\pi_{q'}=1$. We use $V$ to denote the expected payoff, which is a function of $(\pi_{q'})_{q'\in Q\setminus\{q^*\}}$. To prove by contradiction, suppose that the local optimum happens at $\pi_{q'}>0$ for some $q'\neq q_i$. By (20),

$$\frac{\partial}{\partial\pi_{q'}}V-\frac{\partial}{\partial\pi_{q^*}}V\geq 0,$$

but this is a contradiction to $\rho(q,s)\in(\bar\rho_{i-1},\bar\rho_i)$.

## Proof of Proposition 4.2

By continuity of $\bar\rho_i$ for all $i$ and continuity of $\rho(q,s)$ for all $(q,s)$ in $\xi(s)$, for $\xi(s)$ close to 1, $\rho(q_i,s)\in(\bar q_{i-1},\bar q_i)$ by (5). The result then follows immediately from Proposition 4.1.

## Proof of Proposition 4.3

I parameterize the perturbation by letting $\mu_h^L=(\mu_h^H-\varepsilon)(1-\mu)$ and $\mu_{h'}^L=(1-\mu_h^H+\varepsilon)(1-\mu)$. Note that $\mu_{h'}^H=1-\mu_h^H$. We consider $\varepsilon>0$ so that $\xi(h)>\xi(h')$. We denote $\tau(q_i,h;q_i)=\beta_i$ and $\tau(q_i,h';q_i)=\beta_i'$ for $i=1,...,N$. Let $\alpha_j^H=\mu_h^H\beta_j+(1-\mu_h^H)\beta_j'$ and let $\alpha_j^L=(\mu_h^H-\varepsilon)\beta_j+(1-\mu_h^H+\varepsilon)\beta_j'$. By continuity, these are the only relevant deviations. Now, the continuation values can be computed in the same way. In particular, $V_{q_i}(H)$ still takes the form of (14), but with $\alpha_j$ replaced by $\alpha_j^H$, and $V_{q_1}(L)$ still takes the form of (16), but with $\alpha_j$ replaced by $\alpha_j^L$. Moreover, using the same arguments as there, we only need to consider $\beta_1 = \cdots = \beta_N = \beta$ and $\beta'_1 = \cdots = \beta'_N = \beta'$. Thus, the expected payoff from $M^{b,N}(\beta, \beta')$ is now

$$\begin{aligned} F(\beta,\beta') &= p_0 \left[\frac{(1-\alpha^H)(1-\eta)}{1-(1-\eta)\alpha^H}\right]^N u^H \\ &+ (1-p_0)\left\{1-\left[\frac{1-\alpha^L}{1-(1-\mu)(1-\eta)\alpha^L}\right]^N \frac{\eta(1-\eta)^N(1-\mu)^N}{1-(1-\eta)(1-\mu)}\right\} u^L, \end{aligned}$$

where $\alpha^H = \mu_h^H \beta + (1-\mu_h^H)\beta'$ and $\alpha^L = (\mu_h^H - \varepsilon)\beta + (1-\mu_h^H+\varepsilon)\beta'$. Hence, taking derivatives, we have

$$\begin{aligned} \frac{\partial F}{\partial \beta} &= p_0 \left[\frac{1-\alpha^H}{1-(1-\eta)\alpha^H}\right]^{N-1} \frac{(-N)\eta(1-\eta)^N u^H}{[1-(1-\eta)\alpha^H]^2}\mu_h^H \\ &+ (1-p_0)\left[\frac{1-\alpha^L}{1-(1-\mu)(1-\eta)\alpha^L}\right]^{N-1} \frac{N\eta(1-\eta)^N(1-\mu)^N u^L}{[1-(1-\eta)(1-\mu)\alpha^L]^2}(\mu_h^H-\varepsilon), \end{aligned} \tag{21}$$

and

$$\begin{aligned} \frac{\partial F}{\partial \beta'} &= p_0 \left[\frac{1-\alpha^H}{1-(1-\eta)\alpha^H}\right]^{N-1} \frac{(-N)\eta(1-\eta)^N u^H}{[1-(1-\eta)\alpha^H]^2}(1-\mu_h^H) \\ &+ (1-p_0)\left[\frac{1-\alpha^L}{1-(1-\mu)(1-\eta)\alpha^L}\right]^{N-1} \frac{N\eta(1-\eta)^N(1-\mu)^N u^L}{[1-(1-\eta)(1-\mu)\alpha^L]^2}(1-\mu_h^H+\varepsilon). \end{aligned} \tag{22}$$

Now, for any $\varepsilon > 0$, since the first terms in both (21) and (22) are negative but the second terms are positive and since $\mu_h^H/(\mu_h^H-\varepsilon) > 1 > (1-\mu_h^H)/(1-\mu_h^H+\varepsilon)$, $\frac{\partial F}{\partial \beta'} = 0$ implies that $\frac{\partial F}{\partial \beta} < 0$, that is, if optimal $\beta' < 1$, optimal $\beta = 0$. Moreover, this shows that whenever $\delta \geq \delta^b_{K-2}$ and hence at $\alpha^H = \alpha^L = 0$, $\frac{\partial F}{\partial \beta'} \geq 0$ when $\varepsilon = 0$. Thus, for any $\varepsilon > 0$, $\frac{\partial F}{\partial \beta'} > 0$ under $\beta = \beta' = 0$ and hence optimal $\beta' > 0$.

## Proof of Proposition 4.4

Proof of (1). Let $\eta \in (0,1)$ be given. I claim that $M^b_K(\alpha)$ achieves global maximum when $\mu_h^H$ is close to 1. To prove this, I first demonstrate local optimality and then give global arguments. To simplify notation, denote $\mu_h^H$ by $1-\varepsilon$ and $\mu_\ell^L$ by $\mu$, and we can compute the continuation values for $M_K^b(\alpha_1, ..., \alpha_{K-2})$: $V_{q_L}(H) = 0$, $V_{q_L}(L) = u^L$, and

$$V_{q_i}(H) = \left[\prod_{j=i}^{K-2} \frac{1-\alpha_j}{1-(1-\eta)(1-\varepsilon)\alpha_j}\right] [(1-\eta)(1-\varepsilon)]^{K-i-1} u^H; \tag{23}$$

$$V_{q_H}(L) = \frac{(1-\eta)\mu u^L}{1-(1-\eta)(1-\mu)},\ V_{q_H}(H) = \frac{\eta u^H}{1-(1-\eta)(1-\varepsilon)}; \tag{24}$$

$$V_{q_i}(L) = \left\{1 - \left[\prod_{j=i}^{K-2} \frac{1-\alpha_j}{1-(1-\mu)(1-\eta)\alpha_j}\right] \frac{\eta(1-\eta)^{K-i-1}(1-\mu)^{K-i-1}}{1-(1-\eta)(1-\mu)}\right\} u^L \tag{25}$$

By the same arguments as before, it is optimal to set $\alpha_i = \alpha$ for all $i$, and hence

$$F(\alpha) = p_0 \left[\frac{1-\alpha}{1-(1-\eta)(1-\varepsilon)\alpha}\right]^{K-2} [(1-\eta)(1-\varepsilon)]^{K-2} u^H \tag{26}$$

$$+ (1-p_0)\left\{1 - \left[\frac{1-\alpha}{1-(1-\mu)(1-\eta)\alpha}\right]^{K-2} \frac{\eta(1-\eta)^{K-2}(1-\mu)^{K-2}}{1-(1-\eta)(1-\mu)}\right\} u^L \tag{27}$$

The function $F(\alpha)$ is quasi-concave in $\alpha \in [0,1]$, and hence its local optimum is continuous in $\varepsilon$. To see this, note that, disregarding some constants, $F'(\alpha)$ is proportional to

$$\begin{aligned} F'(\alpha) &\simeq -\frac{(1-\alpha)^{N-1}}{[1-(1-\eta)(1-\varepsilon)\alpha]^{N+1}} \left[\frac{\rho_0}{\rho^*}\xi(h)^N\right] + \frac{(1-\alpha)^{N-1}}{[1-(1-\eta)(1-\mu)\alpha]^{N+1}} \\ &= \left[\frac{(1-\alpha)^{N-1}}{[1-(1-\eta)(1-\varepsilon)\alpha]^{N+1}}\right] \left\{\frac{[1-(1-\eta)(1-\varepsilon)\alpha]^{N+1}}{[1-(1-\eta)(1-\mu)\alpha]^{N+1}} - \left[\frac{\rho_0}{\rho^*}\xi(h)^N\right]\right\}, \end{aligned}$$

and it can be verified that the second term in the last equation strictly decreases with $\alpha$ whenever $\varepsilon < \mu$, and hence can have at most one solution to $F'(\alpha) = 0$ besides $\alpha = 1$. Note that $\alpha = 1$ is excluded by the fact that at $\varepsilon = 0$, the optimal SFSA is $M_K^b(\alpha^*)$ with $\alpha^* \in [0,1)$, and hence it is cannot be optimal for $\varepsilon$ small. Now, if $\alpha^* > 0$ then this ensures local optimality. Otherwise, we have $K = N+2$, and one can appeal to Proposition 4.1.

To prove global optimality, let $\varepsilon$ be small so that we have local optimality. Now, consider the set of SFSA $M$ that does not take the form $M_K^b(\alpha)$, denoted by $D$. This set is compact, and, when $\mu_h^H = 1$,

$$W_{M_K^b(\alpha^*)} > \max_{M \in D} W_M. \tag{28}$$

Now, by continuity again, there exists $\varepsilon_1 \in (0, \varepsilon]$ such that for any $\mu_h^H \in [1-\varepsilon, 1]$, (28) still holds.

Proof of (2). Now, let $\mu_h^H > 1-\mu_\ell^L$ be given. As before, I denote $\mu_h^H$ by $1-\varepsilon$ and $\mu_\ell^L$ by $\mu$, and hence $\varepsilon < \mu$.

Proof of (a). Let $N$ be such that $\ln\rho^* - N\ln[\xi(h)] \leq \ln\rho_0 < \ln\rho^* - (N-1)\ln[\xi(h)]$. I claim that, under $M_{N+2}^b$, $\rho(q_i,h) \in (\bar\rho_{i-1},\bar\rho_i)$, $\rho(q_H,\ell) < \bar\rho_0$, and $\rho(q_L,h) < \bar\rho_0$. Then, Proposition 4.1 implies local optimality. First I compute the probabilities under $M_{N+2}^b(\alpha)$, $f(q|\theta)$:

$$\begin{aligned}
f(q_i|H) &= \left[\frac{(1-\eta)(1-\varepsilon)(1-\alpha)}{1-(1-\eta)(1-\varepsilon)\alpha}\right]^{i-1}\frac{\eta}{1-(1-\eta)(1-\varepsilon)\alpha}, \text{ for } i=1,...,N,\\
f(q_H|H) &= \left[\frac{(1-\eta)(1-\varepsilon)(1-\alpha)}{1-(1-\eta)(1-\varepsilon)\alpha}\right]^{N}\frac{\eta}{1-(1-\eta)(1-\varepsilon)},\\
f(q_L|H) &= \frac{(1-\eta)\varepsilon}{1-(1-\eta)(1-\varepsilon)},\\
f(q_i|L) &= \left[\frac{(1-\eta)(1-\mu)(1-\alpha)}{1-(1-\eta)(1-\mu)\alpha}\right]^{i-1}\frac{\eta}{1-(1-\eta)(1-\mu)\alpha}, \text{ for } i=1,...,N,\\
f(q_H|L) &= \left[\frac{(1-\eta)(1-\mu)(1-\alpha)}{1-(1-\eta)(1-\mu)\alpha}\right]^{N}\frac{\eta}{1-(1-\eta)(1-\mu)},\\
f(q_L|L) &= \frac{(1-\eta)\mu}{1-(1-\eta)(1-\mu)}.
\end{aligned}$$

Now, let

$$A \equiv \frac{1-(1-\eta)(1-\mu)}{1-(1-\eta)(1-\varepsilon)}. \tag{29}$$

Then, $A>1$ whenever $\varepsilon<\mu$. And let

$$B(\alpha) \equiv \frac{1-(1-\eta)(1-\mu)\alpha}{1-(1-\eta)(1-\varepsilon)\alpha}. \tag{30}$$

These then imply that

$$\begin{aligned}
\rho(q_i) &= \rho_0\left[\frac{1-(1-\eta)(1-\mu)\alpha}{1-(1-\eta)(1-\varepsilon)\alpha}\right]^i\left(\frac{1-\varepsilon}{1-\mu}\right)^{i-1} = \rho_0 B^i[\xi(h)]^{i-1} \text{ for } i=1,...,N,\\
\rho(q_H) &= \rho_0\left(\frac{1-\varepsilon}{1-\mu}\right)^N\left[\frac{1-(1-\eta)(1-\mu)\alpha}{1-(1-\eta)(1-\varepsilon)\alpha}\right]^N\frac{[1-(1-\eta)(1-\mu)]}{[1-(1-\eta)(1-\varepsilon)]} = \rho_0[\xi(h)]^N B^N A,\\
\rho(q_L) &= \rho_0\frac{\varepsilon[1-(1-\eta)(1-\mu)]}{\mu[1-(1-\eta)(1-\varepsilon)]} = \rho_0\xi(\ell)A,
\end{aligned}$$

Now,

$$\begin{aligned}
\bar\rho_N &= \rho^*\frac{1-(1-\eta)(1-\varepsilon)\alpha}{1-(1-\eta)(1-\mu)\alpha} = \frac{\rho^*}{B},\ \bar\rho_i = \rho^*\frac{1}{[\xi(h)]^{N-i}B^{N-i+1}} \text{ for } i=1,...,N-1,\\
\bar\rho_0 &= \rho^*\frac{1}{[\xi(h)]^N B^N A}.
\end{aligned}$$

Since $N \leq K-2$, optimal $\alpha = 0$ and hence $B = 1$. I verify

$$\rho(q_i)\xi(h) \in (\bar{\rho}_i, \bar{\rho}_{i+1}) \text{ for } i = 1, ..., N; \tag{31}$$

$$\rho(q_H)\xi(\ell) \leq \bar{\rho}_0; \tag{32}$$

$$\rho(q_L)\xi(h) \leq \bar{\rho}_0. \tag{33}$$

To verify (31), which is equivalent to

$$\frac{\rho^*}{[\xi(h)]^{N-i}} \leq \rho_0[\xi(h)]^i \leq \frac{\rho^*}{[\xi(h)]^{N-i-1}},$$

that is,

$$[\xi(h)]^{N-1} \leq \frac{\rho^*}{\rho} \leq [\xi(h)]^N. \tag{34}$$

Moreover, (32) implies (33):

$$\rho_0[\xi(h)]^N A\xi(\ell) \leq \rho^* \frac{1}{[\xi(h)]^N A} \;\Rightarrow\; \rho_0\xi(\ell)A\xi(h) \leq \rho^* \frac{1}{[\xi(h)]^N A}.$$

Now, (32) holds if and only if

$$A^2[\xi(h)]^{2N}\xi(\ell) \leq \frac{\rho^*}{\rho_0}. \tag{35}$$

Now, since $[\xi(h)]^{N+1} \leq [\xi(h)]^{K-1} \leq \xi(\ell) < 1$, (34) implies (35) for $\eta$ large. Note that

$$\lim_{\eta \to 1} A = 1.$$

Proof of (b). In this case, optimal $\alpha = \alpha^* > 0$ solves

$$B^{N+1} = \frac{\rho^*}{\rho_0[\xi(h)]^N}. \tag{36}$$

I verify

$$\rho(q_i)\xi(h) = \bar{\rho}_i \text{ for } i = 1, ..., N; \tag{37}$$

$$\rho(q_H)\xi(\ell) \leq \bar{\rho}_0; \tag{38}$$

$$\rho(q_L)\xi(h) \leq \bar{\rho}_0. \tag{39}$$

To verify (37), which is equivalent to

$$\frac{\rho^*}{[\xi(h)]^{N-i}B^{N-i+1}} = \rho_0 B^i [\xi(h)]^i,$$

which is equivalent to (36).

Moreover, (38) implies (39):

$$\rho_0[\xi(h)]^N B^N A\xi(\ell) \leq \rho^* \frac{1}{[\xi(h)]^N B^N A} \;\Rightarrow\; \rho_0\xi(\ell)A\xi(h) \leq \rho^* \frac{1}{[\xi(h)]^N B^N A}.$$

Now, (38) holds if and only if

$$A^2[\xi(h)]^{2N} B^{2N}\xi(\ell) \leq \frac{\rho^*}{\rho_0}. \tag{40}$$

Now, if we have

$$[\xi(h)]^M \xi(\ell) < 1$$

for some $M \geq N+1$, then there exists $\eta$ large such that (38) holds and $\rho_0\xi(h)^N < \rho^*$ if

$$[\xi(h)]^N < \frac{\rho^*}{\rho_0} < [\xi(h)]^M. \tag{41}$$

Now, local optimality is then implied by Proposition 4.1: in particular, since here $\xi(h) > 1$, the only other deviation is to move to a higher memory state, but this is excluded by (36).

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