Responding to the Inflation Tax*

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Abstract

We adopt mechanism design to study the long-run consequences of inflation on aggregate output, trade, and welfare. Our theory captures multiple channels for individuals to respond to the inflation tax: search intensity (the intensive margin), market participation (the extensive margin), and substitution between money and a higher return asset. To determine the terms of trade in pairwise meetings, we consider socially optimal allocations that are individually rational and immune to pairwise defection. We characterize constrained efficient allocations and show that inflation has non-monotonic effects on both the frequency of trades and the total quantity of goods traded. The model reconciles several qualitative patterns emphasized in empirical macro studies and historical anecdotes, including monetary superneutrality for low inflation rates, non-linearities in trading frequencies, and substitution of money for capital for high inflation rates. While these effects are difficult to capture in previous monetary models, we show how they are intimately related by all being features of an optimal trading mechanism.

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1 Introduction

What are the effects of anticipated inflation on economic activity? Insofar as inflation acts as a tax on cash transactions, the conventional wisdom is that individuals shift consumption away from cash-intensive activities as inflation rises. This channel is present in almost all monetary models and is a key determinant of the welfare cost of inflation (Cooley and Hansen (1989), Lucas (2000)). However, while economists concede that the empirical relationship between inflation and output is markedly different in low inflation environments than in high inflation environments, there is far less consensus on the long-run effects of inflation across monetary models.[1] For instance, in Sidrauski (1967)'s money-in-the-utility-function model, inflation has no real effects while in Stockman (1981)'s cash-in-advance model, inflation has a negative effect on real output for all inflation rates. Since money enters into the economy in an ad-hoc way and there is typically only a single channel for inflation to affect real activity, those monetary models can neither capture the social role of money nor the non-linear effects of inflation documented empirically.

In this paper, we propose a unified monetary model that captures multiple channels for inflation to affect aggregate activity, trade, and welfare. In addition to the effects of inflation on the purchasing power of money and real output, we emphasize additional consequences of inflation that are critical to the functioning of monetary economies. These include (*i*) the effort taken by individuals engaging in market activities to economize on their money holdings, (*ii*) the accumulation of real assets or capital goods that may substitute for money as a means of payment, and (*iii*) the trade and exchange patterns that society adopts. While studies by historians reveal that their consequences for the functioning of monetary economies are both important and severe (Bresciani-Turroni (1931), Bernholz (2003)), we know of no existing study that can capture each of these aspects in a single coherent framework. Indeed there appears to be a disconnect between macroeconomists focusing on long-run aggregates and historians emphasizing micro-level trading behaviors, as the typical competitive markets paradigm can say little about how inflation affects social interactions and trading patterns.

To capture the effects of inflation on individual money holdings and trade, we adopt a framework that has an explicit role for money, without assuming money in the utility function or cash-in-advance constraints. As in Lagos and Wright (2005), our baseline model features alternating rounds of centralized trades and pairwise meetings where a double-coincidence problem and frictions such as limited commitment, no enforcement, and no record-keeping make money essential for trade. As we show, this departure from previous studies matters significantly for capturing the qualitative

[1] Bullard and Keating (1995) find that a permanent increase in anticipated inflation can have a positive effect on real output at low to moderate inflation rates that dissipates at higher levels. Rapach (2003) confirms this non-linear effect in a sample of 14 OECD countries using a structural autoregression framework.

relationships between inflation and economic aggregates that have appeared historically.

To emphasize the different channels for inflation to affect economic activity and trade, we modify the Lagos and Wright (2005) model along three critical dimensions. First, the frequency of trades is determined endogenously by buyers' costly search efforts. In many historical episodes of high inflation, individuals try to reduce the time of carrying money by increasing their frequency of trade, as described by Guttmann and Meehan (1975) for the 1920s hyperinflation in Germany,[2] Heynmann and Leijohnhufvud (1995) and O'Dougherty (2002) for the 1990s high inflations in Latin America, and more recently in Zimbabwe.[3] These historical narratives describe the so-called "hot potato" effect of inflation where individuals expend valuable time and effort trying to spend their money more quickly. We endogenize search effort as a natural way to study this hot potato effect, a phenomenon that has been elusive to capture in previous studies.

Second, we introduce capital goods that can compete with money as a means of payment. While money is typically the primary liquid asset in low inflation environments, societies in times of high inflation tend to use other assets for transactions, such as capital goods. For instance, Bresciani-Turroni (1931) document the effects of capital overaccumulation in Weimar Germany from 1914 to 1923 and observe that "To avoid the effects of the monetary depreciation, German agriculturists continued to buy machines; the 'flight from the mark to the machine,'... was the most convenient and the easiest means of defense against the depreciation of the currency. But towards the end of the inflation, farmers realized that a great part of their capital was sunk in machines, whose number was far above what would ever be needed."[4] To capture this substitution effect, we make no exogenous restrictions on payment arrangements and determine how inflation affects the endogenous choice between money and a higher return asset. While our formalization is stylized, capital goods in our model are meant to capture two critical features of real assets: first, overaccumulation of such assets is inefficient, and second, such assets can serve as payment (or collateral) to facilitate trade. Accordingly, capital goods in our model can be broadly interpreted as any real asset with these two features.

Lastly, a key ingredient of our analysis is the use of mechanism design to determine the terms of trade in pairwise meetings. Following Hu, Kennan, and Wallace (2009), we consider socially

[2]Guttmann and Meehan (1975) describe the experiences of individuals dealing with high inflation in Germany where prices quadrupled each month from 1919 to 1923: "At eleven o'clock in the morning a siren sounded and everybody gathered in the factory forecourt where a five-ton lorry was drawn up loaded brimful with paper money. The chief cashier and his assistants climbed up on top. They read out names and just threw out bundles of notes. As soon as you had caught one you made a dash for the nearest shop and bought just anything that was going."

[3]Wines (2006) summarizes the distortionary effects of high inflation on consumer spending in Zimbabwe: "As soon as [cash] is handed over, its value vanishes: 'As soon as I get it, I have to rush out and spend as much of it as I can... And then there is nothing left for the rest of the month.'"

[4]Another important substitute for money is other currencies that do not depreciate with domestic inflation (Calvo and Vegh (1992)), but we do not pursue that as it would require a model of open economies with monetary exchanges.

optimal allocations that are individually rational and immune to pairwise defection. As in Hu and Rocheteau (2013), we obtain coexistence of money and higher return capital as a feature of the optimal trading mechanism by allowing the proposed allocation to depend both on the buyer's total wealth and portfolio composition. In turn, this approach allows us to consider how inflation affects the endogenous use of assets for transactions and moreover, the trading arrangements implemented by society.[5]

Our results provide a comprehensive picture on the consequences of inflation for all possible inflation rates. In contrast with many previous studies, our analysis pertains to low and moderate inflation economies as well as high inflation economies. As a benchmark, we first consider an economy with money only and hence no production of capital. For a range of low inflation rates, money is superneutral: output and search efforts remain at their efficient levels, irrespective of changes in inflation. When inflation rises above a certain threshold, both the buyer's surplus and search effort increase with inflation, even though the buyer's holdings of real balances falls. Hence for a range of intermediate inflation rates, a rise in inflation leads to a higher frequency of trade but lower output per trade. In addition, we also provide examples where the total quantity of goods traded in the decentralized market (DM) increases with moderate inflation while search efforts are inefficiently high. Our finding that low inflation is costless and becomes socially harmful only with high inflation is broadly consistent with the non-linear relationship between inflation and output documented by Bullard and Keating (1995).[6] Without the presence of capital however, search efforts eventually fall towards zero as inflation tends to infinity. Indeed, as extreme rate of inflation brings monetary trade to near collapse, buyers lose any incentive to search. However, that most economies do not turn into complete barter societies even under hyperinflation suggests that other assets may be used for transactions, a possibility we consider next.

We next introduce capital goods that can compete with money as a means of payment. Following Lagos and Rocheteau (2008), capital goods produced in the centralized market (CM) can be used for production in the next period's CM. For low inflation rates, monetary superneutrality still holds: the capital stock remains at its first-best level irrespective of changes in inflation. Moreover, under a weak sufficient condition, moderate inflation induces search efforts to increase while the capital stock remains at its efficient level. However, overaccumulation of capital is bound to occur as inflation rises, even though search efforts may remain inefficiently high. This result contrasts

[5]As emphasized by Casella and Feinstein (1990), inflation not only affects how individuals economize on their real balances, it also changes the economy's trading patterns: "[H]istorians emphasize hyperinflation's disruptive impact on individuals and on their socioeconomic relationships. Previously stable trading connections were severed, transactions patterns were altered, and normally well-functioning markets collapsed."

[6]In the context of the model, aggregate output or total GDP (that includes both DM and CM outputs aggregated with market prices) is constant for low inflation rates and then declines with higher inflation rates.

dramatically with what happens without capital, where the economy is guaranteed to approach autarky when inflation tends to infinity. Our findings also suggest that capital overaccumulation is a symptom only of high inflation rates, while Tobin effects are small or absent for moderate inflation rates.[7]

This paper proceeds as follows. Section 1.1 relates our findings with the literature. Section 2 presents the baseline environment with endogenous search intensity. Section 3 describes the economy's trading mechanism, and Section 4 characterizes implementable allocations and the effects of inflation on output, search effort, capital accumulation, and welfare. We also consider an alternative formalization where buyers can choose to participate in market activities in lieu of search intensity and analyze the effects of inflation along the extensive margin. Finally, Section 5 closes with concluding remarks. All proofs are in the Appendix.

1.1 Related Literature

A key aspect of our findings is that the hot potato effect and the coexistence of money and higher return capital are both features of an optimal trading mechanism. There are alternative ways to generate the hot potato effect in the literature. In a search model with indivisible money, Li (1994) finds that search intensity can rise with a proxy for inflation, though these results are not robust to relaxing restrictions on the divisibility of money, as discussed by Lagos and Rocheteau (2005). With terms of trade set using competitive search, Lagos and Rocheteau (2005) show that deflation can increase the buyer's surplus and hence search effort even though total surplus decreases, but only for certain parameters and only for inflation rates close to the Friedman rule. In contrast, our results hold more generally across parameterizations and for a range of inflation rates above the Friedman rule.[8] Liu, Wang, and Wright (2011) focus instead on buyers' participation decisions and show that inflation decreases the number of buyers, thereby increasing the frequency of trades. We obtain a similar finding under an optimal trading mechanism in our extension with endogenous entries.

Our formalization of capital is similar to Lagos and Rocheteau (2008) who show that money and capital can coexist only if they have the same rate of return.[9] Regarding the coexistence of

[7] Our finding that individuals first attempt to get rid of their money holdings before substituting with another asset is also consistent with the responses of high inflation documented by Bernholz (2003): "First, the public tries to get rid of the depreciating money when they expect further inflation... and finally, currency substitution sets in and lowers the base, namely the real stock of the inflating money, on which the inflation tax is imposed."

[8] Another way to generate the hot potato effect is to have periodic access to the centralized market, as in Ennis (2009), or to introduce preference shocks, as in Ennis (2008) and Nosal (2011).

[9] Alternative formalizations of capital accumulation in monetary models appear in Shi (1999), Aruoba and Wright (2003), and Aruoba, Waller, and Wright (2011). However, those studies focus on the use of capital goods for production by ruling out its role as a means of payment, which is a key focus of the present paper.

money and higher return assets, Hu and Rocheteau (2013) show that rate-of-return dominance is a feature of good allocations. Our results extend their findings to a model with endogenous search efforts and we find a non-trivial interaction between the two: in the presence of capital, search efforts may remain high as inflation increases while without capital, search efforts eventually fall.

2 Environment

Time is discrete and has an infinite horizon. The economy is populated by a continuum of infinitely-lived agents, divided into a set of *buyers*, denoted by $\mathbb{B}$, and a set of *sellers*, denoted by $\mathbb{S}$. Each date has two stages: the first has pairwise meetings in a decentralized market (called the DM) and the second has centralized meetings (called the CM). Time starts in the CM of period zero.

There is a single perishable good produced in each stage, with the CM good taken as the numéraire. In the CM, all agents have the ability to produce and wish to consume. Agents' labels as buyers and sellers depend on their roles in the DM where only sellers are able to produce and only buyers wish to consume.

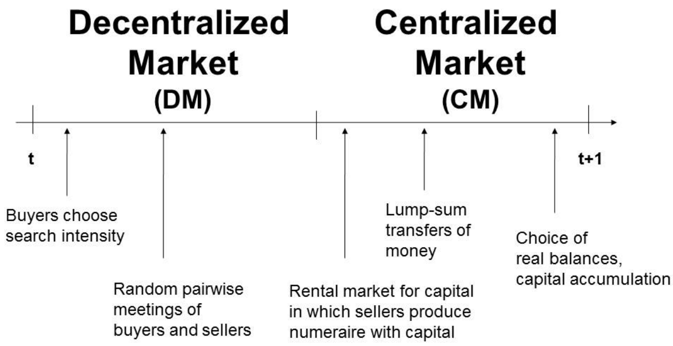
The numéraire good can be transformed into a capital good one for one. Capital goods accumulated at the end of period t are used by sellers at the beginning of the CM of $t+1$ to produce the numéraire good according to the technology $F(k)$, where F is twice continuously differentiable, strictly increasing, strictly concave, and satisfies the Inada conditions $F'(0) = \infty$ and $F'(\infty) = 0$.[10] We also assume that $F'(k)k$ is strictly increasing, strictly concave in k, and has range $\mathbb{R}_+$. Capital goods depreciate fully after one period, and the rental (or purchase) price of capital in terms of the numéraire good at period t is denoted R_t.[11] The assumption of full depreciation is with no loss in generality. For instance, we could have assumed a production technology $f(k)$ and depreciation rate $\delta \in (0, 1)$, and then define $F(k)$ as $F(k) = f(k) + (1 - \delta)k$, which will give us exactly the same analytical results.

There is also an intrinsically useless, perfectly divisible and storable asset called money. Let M_t denote the quantity of money in the end of period-t CM. The relative price of money in terms of the numéraire is denoted ϕ_t. There is an exogenously given gross growth rate of the money supply, which is constant over time and equal to γ; that is, $M_{t+1} = \gamma M_t$. New money is injected if $\gamma > 1$, or withdrawn if $\gamma < 1$, by lump-sum transfers or taxes, respectively. Transfers take place at the

[10]Our model can be reinterpreted as one where buyers are endowed with one unit of labor in the CM and enjoy no utility from leisure, where the CM technology is a constant-return-to-scale neoclassical production function, $G(K, L)$. Our analysis would be unchanged if we replace $F(k)$ with $G(k, 1)$. Although it would complicate the analysis and notations, allowing for elastic labor supply in $G(K, L)$ would not alter our main results. Moreover, our main results go through with a linear production technology for capital, e.g. $F(k) = Ak$ where $A > 0$.

[11]It should be noticed that who operates the technology, F, is irrelevant for our analysis provided that the residual profits, $F(k) - kF'(k)$, are not pledgable in the DM due to lack of commitment.

Figure 1: Timing of Representative Period



beginning of the CM and we specify that they go to buyers only.[12] Lack of record-keeping and private information about individual trading histories rule out unsecured credit, giving a role for money and capital to serve as means of payment. In addition, individual asset holdings are common knowledge in a match.[13] We assume that sellers do not carry real balances or capital across periods. As shown in Hu and Rocheteau (2013), this assumption is with no loss in generality.

Agents are matched in pairs in the DM. We normalize the measure of sellers and buyers each to one. We assume that the seller's search intensity is exogenously given, but buyers can choose their search intensity. At the beginning of the DM, each buyer $b \in \mathbb{B}$ chooses search intensity, $e_b \in [0, 1]$. The average search intensity of buyers is $\overline{e}$, defined as

$$\overline{e} = \int_{b \in \mathbb{B}} e_b db.$$

A buyer exerting effort e to search in the DM incurs cost $\psi(e)$. We assume that for all $e \in [0, 1)$, $\psi(e) \in [0, \infty)$ is twice continuously differentiable, strictly increasing, strictly convex, and satisfies the Inada conditions $\psi(0) = \psi'(0) = 0$, $\lim_{e \to 1} \psi(e) = \infty$, and $\lim_{e \to 1} \psi'(e) = \infty$. Figure 1 summarizes the timing of a representative period.

[12]As our focus is to study the effects of inflation across different inflation rates, the money growth rate is not chosen optimally and is taken as given in the mechanism design problem. To model deflation, the government is assumed to have enough coercive power to collect taxes in the CM, but has no coercive power in the DM.

[13]All our results go through if buyers can hide their asset holdings. This private information problem is secondary for the focus of this paper, and for sake of clarity we choose to ignore it.

Given $\bar{e}$, the number of matches in the DM is determined by a constant-returns-to-scale matching function that depends on *market tightness*, defined as $\theta \equiv 1/\bar{e} \in [1, \infty]$, or the ratio of sellers to the *effective* buyers searching. A high θ implies a thick market for buyers and a thin one for sellers. Given θ, the meeting probability for an individual buyer with search intensity e is $e\alpha(\theta)$ while the meeting probability of a seller is $\alpha(\theta)/\theta$. The function $\alpha(\theta)$ satisfies $\alpha(\theta) \in [0, 1]$ for any $\theta \geq 1$ and is twice continuously differentiable, strictly increasing, strictly concave for $\theta \in [1, \infty)$, and satisfies the Inada conditions $\lim_{\theta\to\infty} \alpha(\theta) = 1$, $\lim_{\theta\to 1} \alpha(\theta) = 0$, $\lim_{\theta\to 1} \alpha'(\theta) \geq 1$, and $\lim_{\theta\to 1} \alpha(\theta)/\theta = 1$.

The instantaneous utility function of a buyer is

$$U^b(q, e, x) = u(q) - \psi(e) + x, \tag{1}$$

where q is consumption in the DM, x is the utility of consuming $x \in \mathbb{R}$ units of numéraire ($x < 0$ is interpreted as production), and e is the buyer's search effort.[14] We assume $u(0) = 0$, $u'(0) = \infty$, $u'(q) > 0$, and $u''(q) < 0$ for $q > 0$. A buyer's lifetime expected utility is $\mathbb{E}_0 \left\{\sum_{t=0}^{\infty} \beta^t U^b(x_t, q_t, e_t)\right\}$, where $\mathbb{E}_0$ is the expectation operator conditional on time-0 information. The discount factor $\beta = (1 + r)^{-1} \in (0, 1)$ is the same for all agents and assumed to be smaller than γ throughout the analysis. Similarly, the instantaneous utility function of a seller is

$$U^s(q, x) = -c(q) + x, \tag{2}$$

where q is production in the DM and x is defined as before. We assume $c(0) = c'(0) = 0$, $c'(q) > 0$, and $c''(q) \geq 0$. Further, we let $c(q) = u(q)$ for some $q > 0$ and denote by q^* the solution to $u'(q^*) = c'(q^*)$. Lifetime utility for a seller is given by $\mathbb{E}_0 \left\{\sum_{t=0}^{\infty} \beta^t U^s(x_t, q_t)\right\}$.

3 Implementation

We study equilibrium outcomes that can be implemented with a *mechanism designer's proposal*. A *proposal* consists of four objects: *(i)* a sequence of functions in the bilateral matches, $o_t : \mathbb{R}^2_+ \to \mathbb{R}^3_+$, each of which maps the buyer's portfolio, (z_t, k_t), into a proposed trade, $(q_t, d_{z,t}, d_{k,t}) \in \mathbb{R}_+ \times [0, z_t] \times [0, k_t]$, where q_t is the DM output produced by the seller and consumed by the buyer, $d_{z,t}$ is the transfer of real balances, and $d_{k,t}$ is the transfer of capital from the buyer to the seller; *(ii)* an initial distribution of money, μ; *(iii)* a sequence of prices for money, $\{\phi_t\}_{t=1}^{\infty}$, and a sequence of

[14] For tractability, the model requires that either the utility of consuming or the cost of producing the CM good is linear. In the formalization here, we simply assume that both CM consumption and production is linear though it would be straightforward to generalize to quasi-linear preferences $U(c) - y$, where c is CM consumption and y is CM production. Agents would then consume c^* in the CM where c^* satisfies $U'(c^*) = 1$. Normalizing $U(c^*) - c^*$ to zero would yield a model equivalent to the one presented here.

rental prices for capital, $\{R_t\}_{t=1}^{\infty}$, both in terms of the numéraire good; *(iv)* a sequence of search intensities of buyers, $\{e_t\}_{t=1}^{\infty}$.

The trading procedure in the DM is given by the following game. Given the buyer's portfolio holding and the associated proposed trade, both the buyer and the seller simultaneously respond with *yes* or *no*: if both respond with *yes*, then the proposed trade is carried out; otherwise, there is no trade. Since both agents can turn down the proposed trade, this ensures that trades are individually rational. We also require any proposed trade to be in the pairwise core.[15] Hence, we only consider trading mechanisms in pairwise meetings that are individually rational and coalition-proof. Agents in the CM trade competitively against the proposed prices, which is consistent with the pairwise core requirement in the DM due to the equivalence between the core for the centralized meeting and competitive equilibria.

We denote s_b as the strategy of buyer $b \in \mathbb{B}$, which consists of three components for any given trading history h^t at the beginning of period t: *(i)* $s_b^{h^t,0}(z,k) = e \in \mathbb{R}_+$ that maps the buyer's portfolio, (z,k), into his search intensity, e, at the beginning of the DM; *(ii)* $s_b^{h^t,1}(z,k) \in \{yes, no\}$ that, contingent on being matched in the DM, maps the buyer's portfolio (z,k) to his *yes* or *no* response in the DM; *(iii)* $s_b^{h^t,2}(z,k,a_b,a_s) \in \mathbb{R}_+^2$ that maps the buyer's original portfolio, (z,k), and the buyer's and seller's choices whether to accept the trade, $a_b, a_s \in \{yes, no\}$, to his final real balances and capital holdings after the CM. The strategy of a seller $s \in \mathbb{S}$ at the beginning of period t consists of a function, $s_s^{h^t,1}(z,k) \in \{yes, no\}$, that represent the seller's response to trade contingent on the buyer's portfolio.

Definition 1. *An equilibrium is a list, $\langle (s_b : b \in \mathbb{B}), (s_s : s \in \mathbb{S}), \mu, \{o_t, \phi_t, R_t, e_t\}_{t=1}^{\infty} \rangle$, composed of one strategy for each agent and the proposal $(\mu, \{o_t, \phi_t, R_t, e_t\}_{t=1}^{\infty})$ such that: (*i*) each strategy is sequentially rational given other players' strategies; and (*ii*) the centralized market clears at every date.*

In what follows, we focus on stationary planner proposals where real balances are constant over time and equilibria such that *(i)* agents follow symmetric and stationary strategies; *(ii)* agents always respond with *yes* in all DM meetings; and *(iii)* the initial distribution of money is uniform across buyers. Following Hu, Kennan, and Wallace (2009), we call such equilibria *simple equilibria*. In a simple equilibrium, $\phi_t = \gamma\phi_{t+1}$ for all t; hence, we can discuss real balances only and leave out ϕ_t from a proposal. Moreover, the proposed DM trades, $o_t(z_t, k_t)$, are the same across time periods and can be written as $o(z,k) = [q(z,k), d_z(z,k), d_k(z,k)]$.

[15]The pairwise core requirement can be implemented directly with a trading mechanism that adds a renegotiation stage as in Hu, Kennan, and Wallace (2009), following the *yes* responses from both agents. The renegotiation stage will work as follows. An agent will be chosen at random to make an alternative offer to the one made by the mechanism. The other agent will then have the opportunity to choose between the two offers.

The outcome of a simple equilibrium is summarized by a list, $(q^p, d_z^p, d_k^p, z^p, k^p, e^p)$, where (q^p, d_z^p, d_k^p) are the terms of trade in the DM, e^p is the buyer's search intensity, and (z^p, k^p) are the portfolio holdings of those buyers. Such an outcome, $(q^p, d_z^p, d_k^p, z^p, k^p, e^p)$, is said to be *implementable* if it is the equilibrium outcome of a simple equilibrium associated with a proposal $\{o, R, e\}$. Given the proposals, we use $\mathcal{CO}(z, k; R)$ to denote the set of allocations in the pairwise core for each (z, k) given a rental price for capital, R.

For a given proposal, o, market thickness, θ, and rental price, R, let $V^b(z, k)$ and $W^b(z, k)$ denote the continuation values for a buyer holding (z, k) upon entering the DM and CM, respectively. Similarly, let $W^s(z, k)$ denote the continuation value for a seller holding (z, k) upon entering the CM. The problem for a buyer in the CM solves

$$W^b(z,k) = \max_{x, \hat{z} \geq 0, \hat{k} \geq 0} \left\{ x + \beta V^b(\hat{z}, \hat{k}) \right\}$$

$$\text{s.t. } x + \gamma \hat{z} + \hat{k} = z + Rk + T$$

where $\hat{z}$ and $\hat{k}$ denote the real balances and capital taken into the next DM, and $T = (M_{t+1} - M_t)\phi_t$ is the lump-sum transfer of money from the government. Since we focus on stationary equilibrium where real balances are constant over time, $\gamma = \frac{\phi_t}{\phi_{t+1}} = \frac{M_{t+1}}{M_t}$. Hence in order to hold $\hat{z}$ real balances in the next period, the buyer must accumulate $\gamma \hat{z}$ units of real balances this period. Substituting $x = z + Rk + T - \gamma \hat{z} - \hat{k}$ from the budget constraint, a buyer's Bellman equation in the CM is

$$W^b(z,k) = z + Rk + T + \max_{\hat{z} \geq 0, \hat{k} \geq 0} \left\{ -\gamma \hat{z} - \hat{k} + \beta V^b(\hat{z}, \hat{k}) \right\}, \tag{3}$$

Due to linear preferences in the CM, note that (i) the buyer's value function is linear in total wealth, $W^b(z, k) = z + Rk + W^b(0, 0)$, and (ii) the maximizing choice of $\hat{z}$ and $\hat{k}$ is independent of the buyer's current wealth, (z, k).

During the first stage, the value function of a buyer with portfolio (z, k) upon entering the DM, $V^b(z, k)$, is given by

$$V^b(z,k) = \max_{e \in [0,1]} \left\{ -\psi(e) + e\alpha(\theta) \left\{ u\left[q(z,k)\right] + W^b\left[z - d_z(z,k), k - d_k(z,k)\right] \right\} + [1 - e\alpha(\theta)] W^b(z,k) \right\}. \tag{4}$$

According to (4), a buyer searching with intensity e meets a seller with probability $e\alpha(\theta)$, consumes $q(z, k)$, and transfers to the seller $d_z(z, k)$ real balances and $d_k(z, k)$ units of capital. The buyer therefore enters the CM with $z - d_z(z, k)$ real balances and $k - d_k(z, k)$ units of capital. With probability $1 - e\alpha(\theta)$, a buyer is unmatched, in which case there is no trade in the DM. Using the linearity of $W^b(z,k)$, (4) simplifies to

$$V^b(z,k) = \max_{e\in[0,1]} \left\{-\psi(e) + e\alpha(\theta)\left\{u\left[q(z,k)\right] - d_z\left(z,k\right) - Rd_k\left(z,k\right)\right\} + W^b(z,k)\right\}. \tag{5}$$

For each portfolio (z,k), we use $e(z,k)$ to denote the optimal search intensity that solves the maximization problem (5). Because ψ is strictly convex, $e(z,k)$ is uniquely defined. Moreover, when $(z,k) = (0,0)$, $e(z,k) = 0$. Noting that as $\theta = 1/e^p$ in equilibrium, the buyer's choice of search intensity, $e^p = e(z^p, k^p)$, solves

$$-\psi'(e^p) + \alpha(1/e^p)\left[u(q^p) - d_z^p - Rd_k^p\right] = 0. \tag{6}$$

Substituting $V^b(z,k)$ with its expression given by (5) into (3), using the linearity of $W^b(z,k)$, and omitting constant terms, the buyer's portfolio problem in the CM can be reformulated as

$$\max_{(z,k)} \left\{-iz - (1+r-R)k - \psi(e(z,k)) + e(z,k)\alpha(\theta)\left\{u\left[q(z,k)\right] - d_z\left(z,k\right) - Rd_k(z,k)\right\}\right\}, \tag{7}$$

where $i = \frac{\gamma-\beta}{\beta}$ is the cost of holding money and $1+r-R$ is the cost of holding capital, which is the difference between the gross rate of time preference and the rental price of capital. Since holding the equilibrium portfolio, (z^p, k^p), is better than $(0,0)$, we must have

$$-iz^p - (1+r-R)k^p - \psi(e^p) + e^p\alpha(1/e^p)[u(q^p) - d_z^p - Rd_k^p] \geq 0. \tag{8}$$

In a similar vein, the Bellman equation for a seller in the CM is

$$W^s(z,k) = z + Rk + \max_{\hat{k}\geq 0} \left\{F(\hat{k}) - R\hat{k}\right\}. \tag{9}$$

According to (9), the seller's choice of input to operate the production technology is such that $F'(\hat{k}) = R$. Due to market clearing in the CM, the aggregate demand for capital must equal the aggregate supply: $k^p = \hat{k}$. Consequently, the equilibrium capital stock, k^p, satisfies

$$F'(k^p) = R \leq 1+r. \tag{10}$$

According to (10), the equilibrium capital stock equates the marginal product of capital, $F'(k^p)$, with the rental rate, R. It is also necessary that $R \leq 1+r$. If $R > 1+r$, buyers will hold an infinite amount of capital, but perfect competition implies that $F'(\infty) = 0 < 1+r < R$, a contradiction.

Using (9), the seller is willing to respond with *yes* to the proposed trade (q^p, d_z^p, d_k^p) only if

$$- c(q^p) + d_z^p + Rd_k^p \geq 0. \tag{11}$$

The above discussion implies that (6), (8), (10), and (11) are necessary conditions to implement an outcome, $(q^p, d_z^p, d_k^p, z^p, k^p, e^p)$. In addition, we also impose the pairwise core requirement. For a given rental price, R, and buyer's portfolio, (z^p, k^p), the pairwise core, denoted by $\mathcal{CO}(z^p, k^p; R)$, is defined as the set of all feasible allocations, $(q, d_z, d_k) \in \mathbb{R}_+ \times [0, z^p] \times [0, k^p]$, such that there are no alternative feasible allocations that would make both parties in the match better off, with at least one of the two being strictly better off. Requiring proposed trades to be in the pairwise core ensures that those trades are coalition proof. A characterization of the pairwise core in a related setting can be found in Hu and Rocheteau (2013)'s Supplementary Appendix B. The following proposition shows that those conditions, together with the pairwise core requirement, are also sufficient.

Proposition 1. *An outcome, $(q^p, d_z^p, d_k^p, z^p, k^p, e^p)$, is implementable if and only if*

$$-iz^p - [1 + r - F'(k^p)]k^p + e^p\alpha(1/e^p)[u(q^p) - d_z^p - F'(k^p)d_k^p] - \psi(e^p) \geq 0, \tag{12}$$

$$d_z^p \leq z^p, \; d_k^p \leq k^p, \tag{13}$$

$$\psi'(e^p) = \alpha(1/e^p)[u(q^p) - d_z^p - F'(k^p)d_k^p], \tag{14}$$

$$-c(q^p) + d_z^p + F'(k^p)d_k^p \geq 0, \tag{15}$$

$$F'(k^p) \leq 1 + r, \tag{16}$$

and $(q^p, d_z^p, d_k^p) \in \mathcal{CO}(z^p, k^p; R)$.

The proof to Proposition 1 is constructive as we explicitly provide the proposed trades to implement the candidate outcome.[16] Proposition 1 extends the implementability result in Hu and Rocheteau (2013). In contrast to their analysis, here we have to worry about the implementation of search efforts, which are determined by the trade surplus that goes to the buyer through (14). As a consequence, the trade surplus to the buyer affects both the portfolio choice and the search effort choice for the buyer. This brings about a new trade-off that is not present in the analysis in Hu and Rocheteau (2013).

[16]The constructed mechanism in the proof of Proposition 1 features a discontinuity in buyers' asset holdings. However, as shown in the Supplemental Material, Section 1, we can make the mechanism continuous.

4 Optimal Allocation

In this section, we study implementable outcomes that are socially optimal. Formally, given an outcome, $(q^p, d_z^p, d_k^p, z^p, k^p, e^p)$, social welfare is defined as the discounted sum of buyers' and sellers' expected utilities:

$$\begin{aligned} \mathcal{W}(q^p, d_z^p, d_k^p, z^p, k^p, e^p) &= -k^p + \lim_{T\to\infty}\sum_{t=1}^{T}\beta^t\left\{e^p\alpha\left(\frac{1}{e^p}\right)[u(q^p)-c(q^p)]-\psi(e^p)+[F(k^p)-k^p]\right\} \\ &= \frac{1}{r}\left\{e^p\alpha\left(\frac{1}{e^p}\right)[u(q^p)-c(q^p)]-\psi(e^p)+[F(k^p)-(1+r)k^p]\right\}. \end{aligned} \tag{17}$$

The first term after the first equality is the utility cost incurred by agents in the initial CM to accumulate the proposed capital stock, k^p; the second term captures the utility flows in subsequent periods and consists of the sum of expected surpluses in pairwise meetings, $e^p\alpha(1/e^p)[u(q^p)-c(q^p)]$, the cost of searching, $\psi(e^p)$, and the output from the production technology net of the depreciated capital stock, $F(k^p)-k^p$.

Definition 2. *An outcome, $(q^p, d_z^p, d_k^p, z^p, k^p, e^p)$, is constrained efficient if it maximizes (17) subject to (12)–(16) and the pairwise core requirement.*

We begin with the benchmark case that maximizes social welfare (17) but without implementability constraints (12)–(16). The solution to this unconstrained problem, which we also call the *first-best allocation*, is given by $q^p = q^*$, $k^p = k^*$, and $e^p = e^*$ that solve

$$u'(q^*) = c'(q^*), \tag{18}$$

$$F'(k^*) = 1 + r, \tag{19}$$

$$\left[\alpha(1/e^*) - \alpha'(1/e^*)/e^*\right][u(q^*) - c(q^*)] = \psi'(e^*). \tag{20}$$

Because (q^*, k^*, e^*) are uniquely determined by these first-order conditions for optimality and because the welfare given by (17) is concave in q and in e (but it is not concave jointly in (q, e)), these necessary conditions are also sufficient. The first-best level of output, q^*, maximizes the match surplus between a buyer and seller, and the first-best level of capital, k^*, ensures that the marginal product of capital compensates for the opportunity cost of holding capital. The first-best level of search intensity, e^*, is derived from the first-order condition on the objective function with respect to e, but taking $q^p = q^*$. Accordingly, the marginal cost of searching, $\psi'(e^*)$, is equal to the corresponding social marginal contribution of searching, $[\alpha(1/e^*) - \alpha'(1/e^*)/e^*]$, times the surplus generated in each trade, $u(q^*) - c(q^*)$.

4.1 Endogenous Search Intensity with Money Alone

Here we consider the economy without the production of capital, that is, we consider constrained-efficient outcomes with the additional constraint $k^p = 0$ (and ignore (16)). The following lemma helps to characterize a constrained-efficient outcome without capital, (q^p, d_z^p, z^p, e^p).

Lemma 1. *Consider an economy with the constraint $k^p = 0$. There exists z^p such that (q^p, d_z^p, z^p, e^p) is a constrained-efficient outcome if and only if the triple (q^p, d_z^p, e^p) solves*

$$\max_{(q,d_z,e)} e\alpha(1/e)[u(q) - c(q)] - \psi(e) \tag{21}$$

subject to

$$-id_z + e\alpha(1/e)[u(q) - d_z] - \psi(e) \geq 0, \tag{22}$$

$$\psi'(e) = \alpha(1/e)[u(q) - d_z], \tag{23}$$

$$-c(q) + d_z \geq 0. \tag{24}$$

Moreover, a solution to (21)– (24) exists, and any solution, (q^p, d_z^p, e^p), satisfies $q^p \leq q^*$, $d_z^p \leq u(q^*)$, and $e^p \leq \hat{e}$, where $\hat{e}$ solves

$$\psi'(\hat{e})/\alpha(1/\hat{e}) = [u(q^*) - c(q^*)]. \tag{25}$$

Because of Lemma 1, we also call the triple (q^p, d_z^p, e^p) a constrained-efficient outcome if it solves (21)–(24). The constraint (22) differs from (12) in that it replaces z^p by d_z^p, and hence implicitly assumes $z^p = d_z^p$. As we show in the proof of Lemma 1, this assumption is satisfied when the first-best is not implementable, and, even when the first-best is implementable, there always exists a constrained-efficient outcome that satisfies this restriction. Moreover, note that there is no pairwise-core requirement in problem (21)–(24). It turns out that when maximizing social welfare, the pairwise-core requirement is not a binding constraint. Accordingly, the set of implementable allocations with respect to Lemma 1 is described by

$$\mathcal{A}^m(i) = \left\{ (q, e) \in \mathbb{R}_+ \times [0, 1] : c(q) \leq u(q) - \frac{\psi'(e)}{\alpha(1/e)} \leq \frac{e\psi'(e) - \psi(e)}{i} \right\}. \tag{26}$$

Note that d_z is pinned down by (23) and hence is not described in $\mathcal{A}^m(i)$. We are now ready to describe constrained-efficient outcomes for the economy with money alone.

Proposition 2. *Consider an economy with the constraint $k^p = 0$. For any $i \geq 0$, a constrained-efficient outcome, $(q^p(i), d_z^p(i), e^p(i))$, exists, and satisfies the following.*

1. *Let*

$$i^* \equiv \frac{e^*\psi'(e^*) - \psi(e^*)}{u(q^*) - \psi'(e^*)/\alpha(1/e^*)} > 0.$$

Then, for all $i \in [0, i^*]$*, the unique constrained-efficient outcome is given by* $(q^p(i), d_z^p(i), e^p(i)) = (q^*, d_z^*, e^*)$*, where* $d_z^* = u(q^*) - \psi'(e^*)/\alpha(1/e^*)$*.*

2. *There exists* $\bar{i} > i^*$ *such that for all* $i \in (i^*, \bar{i}]$*, the unique constrained-efficient outcome,* $(q^p(i), d_z^p(i), e^p(i))$*, satisfies* $q^p(i) < q^*$*,* $d_z^p(i) < d_z^*$*, and* $\frac{d}{di}e^p(i) > 0$*.*

3. *For any* $e \in (0, 1]$*, there exists* $i_e > i^*$ *such that if* $i > i_e$*, then any constrained-efficient outcome,* $(q^p(i), d_z^p(i), e^p(i))$*, satisfies* $q^p(i) < q^*$*,* $d_z^p(i) < d_z^*$*, and* $e^p(i) < e$*.*

4. *Suppose that* $\psi''(0) \in (0, \infty)$ *and that for some* $\delta > 0$*,* $\lim_{q\to 0}(c^{-1} \circ u)'(q)q^{0.5+\delta} > 0$*. Then, for any* i*, any constrained-efficient allocation under* i *has* $d_z^p(i) > 0$*, and has a positive welfare.*

While the proof of Proposition 2 (1) only requires verifying constraints (22)–(24), the proof of Proposition 2 (2) is non-standard since the constraint set is not convex and the objective function is not globally concave. Instead, we employ the Implicit Function Theorem to find a solution to the first-order conditions and use continuity to establish that the solution is also a global maximizer. While we cannot give an explicit expression for the upper bound on the inflation rate below which search intensity increases, we later provide numerical examples to quantify this threshold.

Proposition 2 summarizes the effects of inflation in a pure monetary economy in terms of three inflationary regimes. According to Proposition 2 (1), the highest nominal interest rate for implementing the first best is strictly positive: $i^* > 0$. Hence, the Friedman rule, defined as $i = 0$, is sufficient but not necessary to achieve maximal welfare.[17] For all $i \in [0, i^*]$, money is superneutral and all welfare-relevant variables are at their first-best levels. While this superneutrality result also appears in Hu, Kennan, and Wallace (2009), a notable difference here is that the first best cannot be implemented by giving all the surplus to buyers with the equilibrium amount of real balances.[18] If this were the case, then under the first-best level of output, search intensity would be given by (23) with $q = q^*$ and $d_z = c(q^*)$, and hence equal to $\hat{e}$ given by (25). But due to search externalities, $\hat{e} > e^*$: by (20),

$$\psi'(e^*)/\alpha(1/e^*) < \psi'(e^*)/[\alpha(1/e^*) - \alpha'(1/e^*)/e^*] = [u(q^*) - c(q^*)] = \psi'(\hat{e})/\alpha(1/\hat{e}).$$

[17]This finding differs from the typical result in monetary models that rely on exogenously given trading mechanisms such as pairwise bargaining. There, the Friedman rule is typically necessary for efficiency, at least with regards to the amount of output traded in a match. With endogenous participation or entry however, the Friedman rule need not be optimal. See also Rocheteau and Wright (2005) and Berentsen, Rocheteau, and Shi (2007) for a related discussion.

[18]As in Hu, Kennan, and Wallace (2009), the optimal mechanism punishes buyers with lower than equilibrium amount of real balances by giving them lower surpluses.

To discourage buyers from over-searching, the optimal mechanism gives buyers a fraction of the surplus while the seller's participation constraint, (24), is not binding at the optimum.

Second, for moderate to high inflation rates, the first-best allocation is no longer incentive feasible: both DM output and search intensity deviate from their first-best levels when $i \in (i^*, \bar{i}]$. For nominal interest rates in this range, inflation increases the buyer's search effort and hence the frequency of trades. While this result resembles the so-called "hot potato" effect of inflation, the underlying mechanism in our model differs from the conventional rationale. The standard explanation is that higher inflation itself induces buyers to search harder in order to get rid of their money holdings faster. However, this reasoning implicitly assumes a cash-in-advance constraint without which buyers may not hold cash in the first place. Instead, in our setting, the optimal mechanism dictates buyers to have more surplus in equilibrium as inflation rises above i^*, thereby inducing buyers to search harder.

The intuition for why the optimal mechanism prescribes both the buyer's surplus and search effort to increase with inflation can be seen from Figures 2 and 3, which depict the implementable set, $\mathcal{A}^m(i)$, at $i = i^*$ and how it changes with an increase in i. In Figure 2, the first-best allocation, (q^*, e^*), lies on the boundary of the lower curve, which corresponds to the buyer's participation constraint, (22), being binding, but lies strictly below the upper curve, which corresponds to the seller's participation constraint, (24), being slack. Figure 3 shows that as the nominal interest rate increases from i^* to $i' > i^*$, the buyer's constraint shifts upward while the seller's constraint is not affected. As the objective function, (21), is locally concave, the constrained-efficient level of search intensity increases with inflation. Moreover, because any output higher than q^* would violate the pairwise-core requirement, Lemma 1 implies that output per trade falls as inflation increases.

The above argument concerning the rise in search intensity is valid for nominal interest rates near i^*. Indeed, Proposition 2 (3) shows that search intensity can be arbitrarily small when the inflation rate is sufficiently high. This may be seen from Figure 3: as inflation increases, the buyer's constraint shifts leftward and, for high inflation rates, the implementable output must fall to an arbitrarily small amount. Consequently, the economy will eventually collapse into autarky as inflation rises.

Here we provide numerical examples that illustrate Proposition 2 and quantify the threshold for search effort to increase with inflation. We consider a family of fairly standard functional forms: $u(q) = \frac{(q+b)^{1-\sigma} - b^{1-\sigma}}{1-\sigma}$, $c(q) = \frac{q^\kappa}{\kappa}$, $\psi(e) = c\left(\frac{e}{1-e}\right)^\rho$, and $\alpha(\theta) = 1 - \exp(1-\theta)$ where $\theta = 1/e$. We set $b = 0.0001$, $c = 0.4$, $\rho = 2$, $r = 0.02$, $\sigma = 0.7$, and report results for different values of κ.[19]

[19]While we do not present calibrated examples, we do investigate the sensitivity of our results to changes in parameters. Changing c or r does not affect much the main qualitative results, though obviously does affect e.g. the magnitude for the threshold nominal interest rate, i^*. The examples are most sensitive to different values for σ, which controls the concavity DM utility function, and κ, which controls the convexity of the DM cost function.

Figure 2: Implementable Set, $\mathcal{A}^m(i^*)$

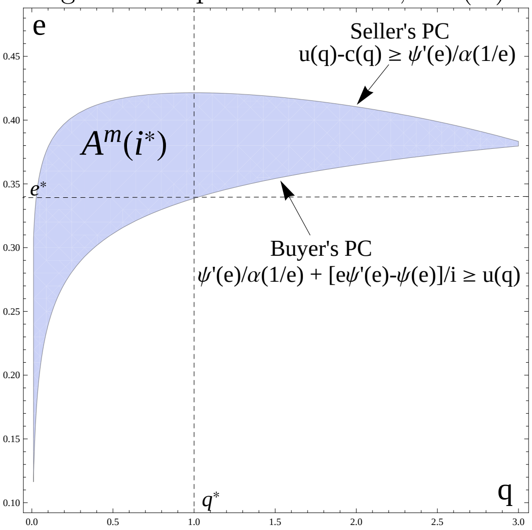


Figure 3: Implementable Set Shrinks As $i \uparrow$

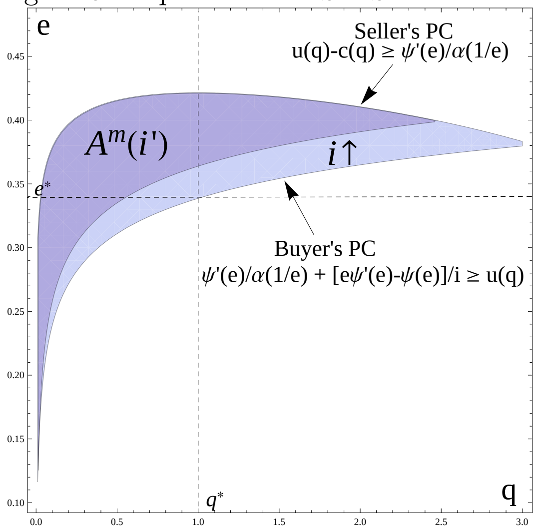


Figure 4: Output per Match

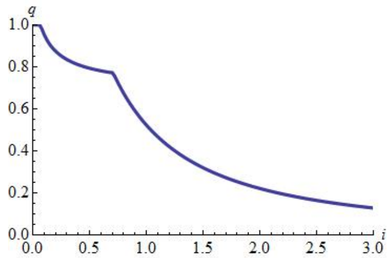
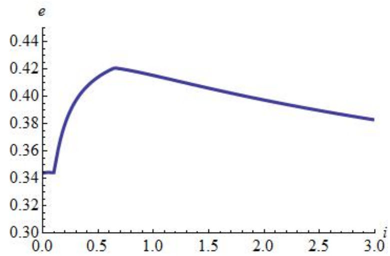


Figure 5: Search Intensity



In what follows, we define DM aggregate output as the total quantity of goods traded, or production in the decentralized market:

$$\mathcal{Q} \equiv \underbrace{e\alpha(1/e)}_{\text{matching prob.}} \times \underbrace{q}_{\text{output per trade}} . \tag{27}$$

Figures 4–9 assume $\kappa = 1$ and plot output per match, search effort, DM aggregate output, the buyer's matching probability, real balances, and the buyer's surplus for a range of nominal interest rates. In these examples, the threshold nominal interest rate below which the first-best is implementable is given by $i^* = 0.09$. Assuming each period corresponds to a year, this corresponds to a threshold inflation rate of $\gamma^* - 1 = (1 + r)^{-1}(1 + i^*) - 1 = 0.07$, or 7% annual inflation, below which money is superneutral.

Figures 5 and 9 show that the buyer's search effort and buyer's surplus move in the same

Figure 6: Aggregate Output

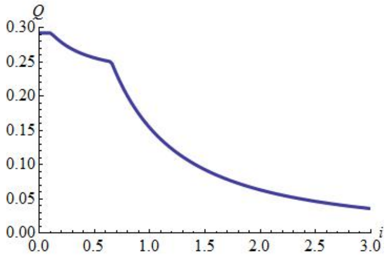


Figure 7: Matching Probability

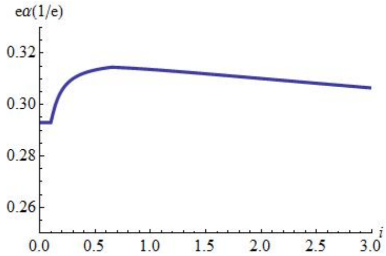


Figure 8: Real Balances

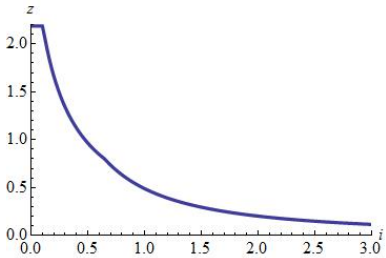
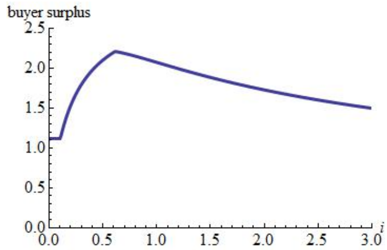


Figure 9: Buyer's Surplus



direction for $i > i^*$, both reaching their maximum at $i = \bar{i}$, as suggested by Proposition 2 (2) and (3). Note also that $\bar{i}$ can be fairly high, corresponding to $\bar{i} = 0.65$ in the examples above, or 63% annual inflation. However, Proposition 2 (3) implies that search efforts eventually fall towards zero as inflation tends to infinity.

We also find examples where DM aggregate output increases with a range of moderate inflation rates. This can be seen in Figure 10 which plots the total quantity traded in the DM, $\mathcal{Q} = e\alpha(1/e)q$, assuming $\kappa = 5$. When $i \in (i^*, \bar{i}]$, our model has two opposing effects: search intensity and hence the frequency of trades, $e\alpha(1/e)$, increases with inflation while DM quantity traded per match, q, falls with inflation. In our examples, we find that the responsiveness of DM output to inflation is decreasing in the parameter κ, so that output is less responsive to inflation when $c(q)$ is more convex. Hence when κ is relatively large, it is possible for the total quantity traded, $e\alpha(1/e)q$, to go up with inflation. Nevertheless, total GDP in our economy, defined as the sum of both DM and CM outputs aggregated using the implicit price in the DM, is proportional to real balances and hence decreases with inflation whenever the first-best is not implementable.

Our non-monotonicity results on search efforts and quantity traded contrast sharply with many previous studies that study endogenous search decisions under pairwise bargaining. In particular, Lagos and Rocheteau (2005) show that under Nash bargaining, the buyer's search effort falls monotonically with inflation. As this trading protocol is held constant for different inflation rates,

Figure 10: Aggregate Output

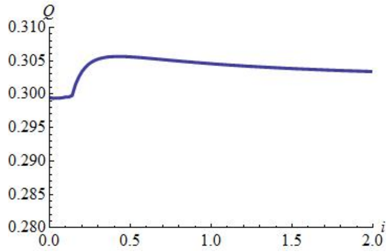
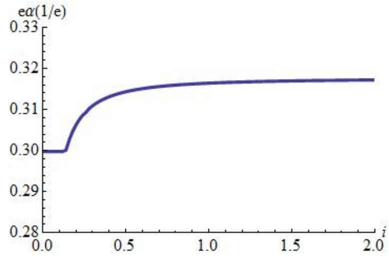


Figure 11: Matching Probability



both the buyer's real balances and surplus fall with inflation, and hence search efforts fall as well.[20] There is another key difference to most literature on search intensity when we introduce capital goods. In most papers, as in our model with money alone, the rise in search intensity, if any, would eventually revert as inflation rises. This result seems at odds with observations describing "hot potato" behavior in economies with high inflation. However, while under competitive search there cannot be rate-of-return dominance for money and capital, we can overturn this eventual collapse when we introduce an alternative means of payments that does not depreciate with inflation, a critical feature of our model that we turn to next.

4.2 Endogenous Search Intensity with Money and Capital

Here we study constrained-efficient outcomes when both money and capital are present in the economy. Since the capital good may also be used as a medium of exchange, we will show that the economy may never collapse into autarky even for very high inflation rates. Less obvious is the extent to which search intensity changes with inflation in the presence of capital, and we obtain a sufficient condition under which the buyer's search intensity increases with moderate inflation while the capital stock remains at its first-best level. However, in contrast to the model with money alone, there is no general guarantee that search intensity is bound to decrease as inflation tends to infinity in the presence of capital.

Before considering the economy with both money and capital, it is useful to first consider the economy with capital as the only liquid asset. This allows us to determine what is achievable with capital alone. Imposing the additional constraint $z = 0$, an outcome may be denoted by (q, d_k, k, e).

Lemma 2. *Consider an economy with the constraint $z = 0$. A constrained-efficient outcome,*

[20] However, they find that search intensity can rise with inflation under competitive search, but only for certain parameterizations and only for inflations rates close to the Friedman rule. In contrast, we find that search intensity rises with inflation holds more generally for a range of positive inflation rates and is a feature of the optimal mechanism for pairwise meetings.

(q^c, d_k^c, k^c, e^c), exists. Moreover, the first-best is implementable if and only if

$$(1+r)k^* \geq u(q^*) - \frac{\psi'(e^*)}{\alpha(1/e^*)}.$$

When the first-best is not implementable, $d_k^c = k^c > k^*$. Moreover, the maximal social welfare given by

$$\mathcal{W}^c = \frac{1}{r}\left\{e^c\alpha(1/e^c)[u(q^c) - c(q^c)] - \psi(e^c) + F(k^c) - (1+r)k^c\right\}$$

is strictly greater than what is achievable with the additional constraint that $k = k^*$, denoted $\mathcal{W}^0$.

Lemma 2 implies that the first-best allocation is implementable without money when the first-best capital stock, k^*, is sufficiently large. In that case, the aggregate capital stock is sufficiently abundant to allow buyers to finance consumption of the first-best. Since the first-best is implementable with $z = 0$, money is not essential.

When instead k^* is insufficient to meet the economy's liquidity needs, the optimal mechanism features an overaccumulation of capital ($k^c > k^*$) in the absence of money. In addition, quantities traded in the DM are inefficiently low ($q^c < q^*$). With a shortage of liquidity, society faces a trade-off between two inefficiencies, as highlighted by Hu and Rocheteau (2013): (*i*) the shortage of capital for liquidity purposes, and (*ii*) the overaccumulation of capital for productive purposes. Lemma 2 then shows that, whenever the first best is not implementable, overaccumulation of capital is socially optimal in order to mitigate the shortage of liquidity (without the presence of money). Note that since it is always feasible to set $z = 0$, $\mathcal{W}^c$ gives a lower bound on welfare when both money and capital are present.

For comparison later on, we provide some numerical examples given in Table 1 for the economy with capital alone. These examples assume the functional forms given in the previous subsection plus $F(k) = Ak^a + (1-\delta)k$. We set $b = 0.0001$, $c = 0.4$, $\rho = 2$, $\kappa = 1$, $r = 0.02$, $a = 0.3$, $A = 0.8$, $\delta = 0.8$, and consider two values of σ, 0.3 and 0.7. In both cases, the first-best is not implementable and there is overaccumulation of capital. However, when $\sigma = 0.3$, equilibrium search intensity is lower than the first-best level; for $\sigma = 0.7$, search intensity is higher than the first-best.

Table 1: Constrained-Efficient Outcomes with Capital Alone

	First-Best	$\sigma = 0.3$	First-Best	$\sigma = 0.7$
Output	$q^* = 1$	$q = 0.29$	$q^* = 1$	$q = 0.32$
Search Effort	$e^* = 0.22$	$e = 0.18$	$e^* = 0.34$	$e = 0.41$
Capital	$k^* = 0.17$	$k = 0.32$	$k^* = 0.17$	$k = 0.37$

We now turn to the case where both money and capital can serve as media of exchange. The next proposition summarizes the effects of inflation on implementable allocations when there is a shortage of capital. To simplify notation, we call a tuple $(q^p(i), z^p(i), k^p(i), e^p(i))$ a constrained-efficient outcome under a nominal interest rate i if there exists $(d_z^p, d_k^p) \le (z^p(i), k^p(i))$ such that $(q^p(i), d_z^p, d_k^p, z^p(i), k^p(i), e^p(i))$ maximizes social welfare, (17), subject to the implementability constraints, (12)–(16). As in the previous section, the pairwise core requirement is not binding for the constrained-efficient outcome.

Proposition 3. *Suppose* $(1+r)k^* < u(q^*) - \psi'(e^*)/\alpha(1/e^*)$. *For any* $i \ge 0$, *a constrained efficient outcome,* $(q^p(i), z^p(i), k^p(i), e^p(i))$, *exists, and satisfies the following.*

1. *Let*
$$i^{**} = \frac{e^*\psi'(e^*) - \psi(e^*)}{u(q^*) - (1+r)k^* - \psi'(e^*)/\alpha(1/e^*)} > i^*.$$
For all $i \in [0, i^{**}]$, *the constrained-efficient outcome,* $(q^p(i), z^p(i), k^p(i), e^p(i))$, *is unique (except for* $z^p(i)$*) , and satisfies* $q^p(i) = q^*$, $z^p(i) > 0$, $k^p(i) = k^*$, *and* $e^p(i) = e^*$.

2. *Suppose* $1 + r + F''(k^*)k^* < \frac{-F''(k^*)k^*}{i^{**}}$. *There exists* $\bar{i}' > i^{**}$ *such that for all* $i \in (i^{**}, \bar{i}']$, *the unique constrained-efficient outcome,* $(q^p(i), z^p(i), k^p(i), e^p(i))$, *satisfies* $q^p(i) < q^*$, $k^p = k^*$, $z^p(i) > 0$, *and* $e^p(i) > e^*$. *Moreover,* $e^p(i)$ *is strictly increasing in* $i \in [i^{**}, \bar{i}']$.

3. *There exists* $\hat{i}$ *such that, for each* $i > \hat{i}$, *and for each constrained-efficient outcome, we have* $k^p(i) > k^*$. *Moreover,* $z^p(i) \to 0$ *as* $i \to \infty$ *but maximum welfare converges to* $\mathcal{W}^c > \mathcal{W}^0$.

Proposition 3 assumes $(1 + r)k^* < u(q^*) - \psi'(e^*)/\alpha(1/e^*)$ to allow a role for money. Proposition 3 (1) shows that, similar to the pure monetary economy studied previously, the first-best is implementable for all $i \in [0, i^{**}]$. In this range, money is superneutral, in which case changes in inflation has no real effects on output or the capital stock.

For a range of intermediate inflation rates, Proposition 3 (2) gives a sufficient condition under which inflation has no effects on the capital stock even though output is inefficiently low and search intensity is inefficiently high. For instance, when $F(k) = Ak^a$, the sufficient condition holds if a is not too large or i^{**} is relatively small. We remark here that money is essential for a range of nominal interest rates above i^{**}, even without imposing the sufficient condition, $1+r+F''(k^*)k^* < \frac{-F''(k^*)k^*}{i^{**}}$ (see Claim 2 in the proof of Proposition 3 (2)). As in Hu and Rocheteau (2013), we obtain rate-of-return dominance whenever both money and capital are used as media of exchange.

Proposition 3 (3) shows that capital overaccumulation is bound to occur as inflation rises, even without the sufficient condition in part (2). In turn, the monetary sector eventually collapses. In contrast to the case with money alone, however, the economy never collapses into autarky as capital can always be used as a medium of exchange, and search intensity can remain inefficiently high even at high inflation rates. Indeed, as welfare converges to the level where only capital is the medium of exchange, $\mathcal{W}^c$, Proposition 3 (3) suggests that search intensity also converges to its level without money, which may be higher or lower than e^*.

Figure 12: Output per Match

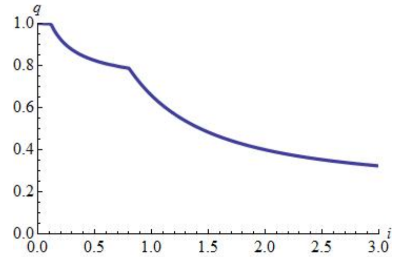


Figure 13: Search Intensity

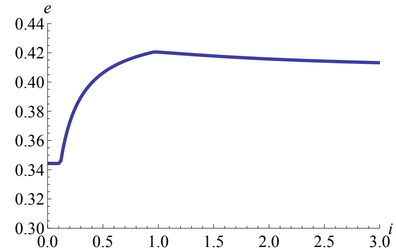


Figure 14: Aggregate Output

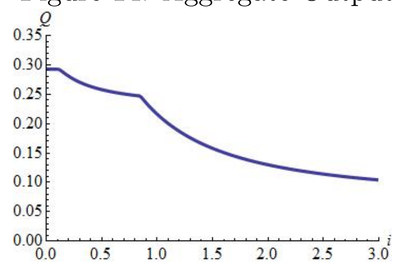


Figure 15: Matching Probability

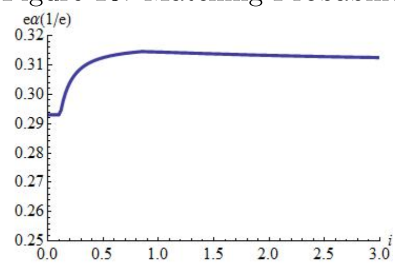


Figure 16: Real Balances

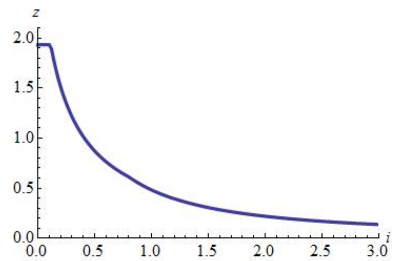
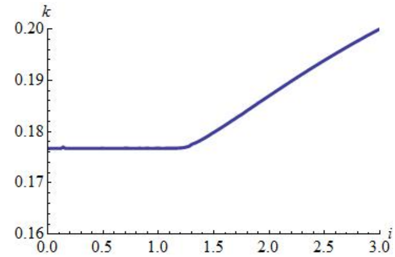


Figure 17: Capital



Figures 12–17 are numerical examples illustrating our findings in Proposition 3. We assume the same functional forms as before with $b = 0.0001$, $c = 0.4$, $\rho = 2$, $\kappa = 1$, $r = 0.02$, $a = 0.3$, $A = 0.8$, $\delta = 0.8$, and $\sigma = 0.7$. Figure 13 plots the buyer's search intensity and shows that search intensity remains above its first-best level and approaches the value reported in Table 1 for the economy with capital alone. As can be seen from Figure 13, the rise in search intensity can persist even for high inflation rates. These results are consistent with many recorded historical episodes

of the “hot potato” effect of inflation as described in Bresciani-Turroni (1931), Heynmann and Leijohnhufvud (1995), and O’Dougherty (2002). For high inflation rates, the constrained-efficient allocation features a substitution effect of inflation where the optimal mechanism prescribes buyers to substitute money for capital as inflation increases, as can be seen in Figures 16 and 17. This Tobin effect turns out to be an optimal way of responding to inflation as doing so allows agents to maintain consumption in the DM even as inflation gets very high, as can be seen in Figure 12.

Finally, our previous finding that DM aggregate output can rise with inflation also carries over to the model with both money and capital. Figure 18 plots aggregate output in the DM, $\mathcal{Q} \equiv e\alpha(1/e)$, assuming $\kappa = 5$. As before, when output per trade is relatively unresponsive to inflation, it is possible for moderate inflation to induce an overall increase in the total number of DM trades.

Figure 18: Aggregate Output

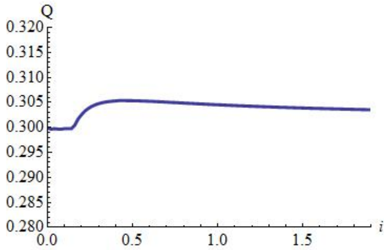
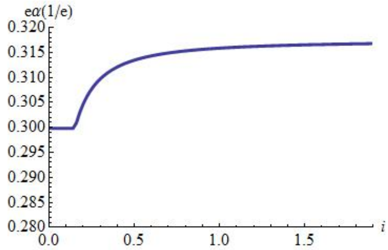


Figure 19: Matching Probability



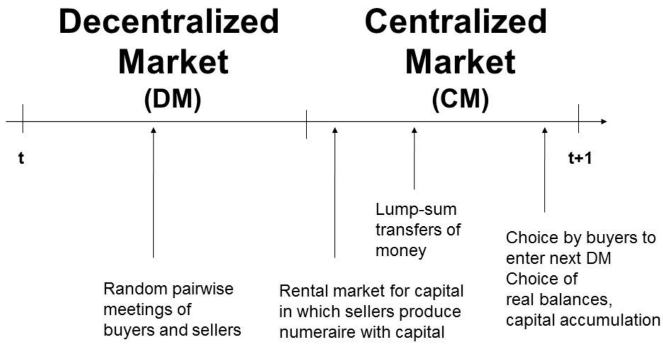
4.3 Endogenous Participation

Here we consider an endogenous participation decision and study the effects of inflation along the extensive margin, similar to Liu, Wang, and Wright (2011).[21] We modify our baseline environment as follows. Instead of choosing search intensity, buyers can choose whether or not to enter the DM each date before the DM opens. If a buyer decides to enter, he incurs a fixed cost $\upsilon > 0$ for doing so. We assume that this entry decision is made together with the portfolio decision in the previous period’s CM. In any case, the buyer must take into account their entry decision when making their portfolio decision. Sellers enter for free, and we assume that a unit measure of sellers always enter each period. The timing of a representative period is summarized in Figure 20.

Given the measure of buyers entering the DM, denoted n, market thickness is given by $\theta = 1/n$, and the buyer’s matching probability is given by $\alpha(\theta)$, where the function α satisfies the assumptions given in Section 2. Under the Inada conditions on α, we may assume that $n \in [0, 1]$. We also modify

[21] Rocheteau (2012) considers a similar endogenous participation decision where agents can choose to be buyers or sellers in the DM. That formalization leads to similar results to the ones presented here.

Figure 20: Timing of Representative Period (Endogenous Participation)



the production technology using the capital good and assume that $F(k) = Ak$ with $A < 1+r$. This modification greatly simplifies the analysis and avoids issues such as whether or not buyers who do not enter the DM hold capital because the efficient amount of capital is zero.

As in Section 3, we study simple equilibria that can be implemented by a mechanism designer's proposal. Here, a proposal consists of (μ, o, ϕ, R, n), where the only new element is the proposed proportion of buyers entering the DM. The trading protocols are defined as before and the strategies and simple equilibria can be defined as before. An outcome then consists of $(q^p, d_z^p, d_k^p, z^p, k^p, n^p)$. We call such an outcome *implementable* if it is the equilibrium outcome of a simple equilibrium associated with a planner's proposal.

For a given proposal, o, market thickness, θ, and rental price, R, the value function of a buyer who decides to enter the DM and who holds (z, k) upon entering the DM, $V^b(z, k)$, is given by

$$V^b(z,k) = \alpha(\theta)\left\{u\left[q(z,k)\right] + W^b\left[z - d_z(z,k), k - d_k(z,k)\right]\right\} + [1-\alpha(\theta)]W^b(z,k) - v, \tag{28}$$

and the value function of a buyer with (z, k) upon entering the CM, $W^b(z, k)$, solves

$$W^b(z,k) = z + Rk + \max\left\{\beta W^b(0,0), \max_{\hat{z}\geq 0, \hat{k}\geq 0}\left\{-\gamma\hat{z} - \hat{k} + T + \beta V^b(\hat{z},\hat{k})\right\}\right\}. \tag{29}$$

We can then further simplify (28) and reformulate the buyer's portfolio problem in the CM as

$$\max\left\{0, \max_{(z,k)}\left\{-iz-(1+r-R)k-\alpha(\theta)\left\{u\left[q(z,k)\right]-d_z\left(z,k\right)-Rd_k(z,k)\right\}-v\right\}\right\}, \tag{30}$$

where $i=\frac{\gamma-\beta}{\beta}$. In equilibrium, free entry of buyers implies that the above maximization problem in (30) should end up with a tie, that is, if $(q^p, d_z^p, d_k^p, z^p, k^p, n^p)$ is an equilibrium outcome, then

$$-iz^p-(1+r-R)k^p+\alpha(1/n^p)[u(q^p)-d_z^p-Rd_k^p]-v=0. \tag{31}$$

Finally, following the same reasoning as in Section 3, we can conclude that in equilibrium, $R=A$. The following lemma characterizes implementable outcomes.

Lemma 3. *An outcome, $(q^p, d_z^p, d_k^p, z^p, k^p, n^p)$, with $n^p \in (0,1)$ is implementable if and only if*

$$-iz^p-(1+r-A)k^p+\alpha\left(1/n^p\right)\left[u(q^p)-d_z^p-Ad_k^p\right]-v=0, \tag{32}$$

$$d_z^p \le z^p,\ \ d_k^p \le k^p, \tag{33}$$

$$-c(q^p)+d_z^p+Ad_k^p \ge 0, \tag{34}$$

and $(q^p, d_z^p, d_k^p) \in \mathcal{CO}(z^p, k^p; A)$.

Given an outcome, $(q^p, d_z^p, d_k^p, z^p, k^p, n^p)$, social welfare is defined as the discounted sum of buyers' and sellers' expected utilities:

$$\mathcal{W}(q^p, d_z^p, d_k^p, z^p, k^p, n^p) \quad = \frac{1}{r}\left\{n^p\alpha\left(\frac{1}{n^p}\right)[u(q^p)-c(q^p)]-n^pv+n^p[Ak^p-(1+r)k^p]\right\}. \tag{35}$$

We say that an outcome is *constrained efficient* if it maximizes (35) subject to (32)–(34) and the pairwise core requirement. Using a similar logic as Lemma 2, we can restrict attention to outcomes where buyers spend all their asset holdings in pairwise meetings and ignore the pairwise-core requirement. Hence, we only look for outcomes of the form (q^p, z^p, k^p, n^p). The first-best level of output, capital, and measure of buyers entering maximize (35) and is given by $(q^*, 0, n^*)$, where $u'(q^*)=c'(q^*)$ and

$$\left[\alpha\left(1/n^*\right)-\alpha'\left(1/n^*\right)/n^*\right]\left[u(q^*)-c(q^*)\right]=v. \tag{36}$$

Throughout this section we assume $\alpha(1/n^*)[u(q^*)-c(q^*)]>v$, since otherwise, the buyer is not willing to participate in the DM even at the first-best arrangement.

We remark that a constrained efficient outcome exists under the additional constraint that $z^p=0$ and the optimal value for welfare is unique, denoted by $\mathcal{W}^c$. $\mathcal{W}^c$ may be zero. It will also be useful to define a threshold for the fixed cost of entering the DM,

$$\bar{v} = \frac{\alpha(1/n^*)[c(q^*)(u(\bar{q}) - c(\bar{q})) - c(\bar{q})(u(q^*) - c(q^*))]}{c(q^*) - c(\bar{q})} > 0,$$

where $\bar{q} < q^*$ solves $u'(\bar{q})/c'(\bar{q}) = (1 + r)/A$. We also define $\bar{A} = 1/[(1 + r)(1 + i^*)]$.

Proposition 4. *For any $i \geq 0$, a constrained efficient outcome, $(q^p(i), z^p(i), k^p(i), n^p(i))$, exists, and satisfies the following.*

1. *Let $i^* = \frac{\alpha(1/n^*)[u(q^*) - c(q^*)] - v}{c(q^*)} > 0$. Then, for all $i \in [0, i^*]$, the constrained-efficient outcome, $(q^p(i), z^p(i), k^p(i), n^p(i))$, is unique, and satisfies $q^p(i) = q^*$, $z^p(i) \geq c(q^*)$, $k^p(i) = 0$, and $n^p(i) = n^*$.*

2. *Suppose $v < \bar{v}$ or $A < \bar{A}$. There exists $\bar{i} > i^*$ such that for all $i \in (i^*, \bar{i}]$, the unique constrained-efficient outcome, $(q^p(i), z^p(i), k^p(i), n^p(i))$, satisfies $q^p(i) < q^*$, $z^p(i) = c(q^p)$, $k^p(i) = 0$, and $n^p(i) < n^*$. Moreover, $n^p(i)$ is strictly decreasing in $i \in (i^*, \bar{i}]$.*

3. *Suppose $A = 0$. There exists $\bar{i}$ such that $i > \bar{i}$ implies that the constrained-efficient outcome is autarky.*

4. *Suppose $\mathcal{W}^c > 0$. Then, if $i > \bar{i}$, any constrained-efficient outcome, $(q^p(i), z^p(i), k^p(i), n^p(i))$, satisfies $k^p(i) > 0$.*

When $A = 0$, there is no capital production in the economy and hence is a special case of Proposition 4. For this case and for cases $A < \bar{A}$, the findings in Proposition 4 resemble some aspects of Proposition 2. In both cases, the first-best is achievable for a range of low inflation rates, and the buyer's matching probability ($e\alpha(1/e)$ for Proposition 2 and $\alpha(1/n)$ here) increases with inflation for a range of intermediate inflation rates. With endogenous participation, this result is also similar to Liu, Wang, and Wright (2011) and captures the idea that inflation induces individuals to trade more quickly. Note that we obtain the first best for a range of low inflation rates, while the first best only occurs as a knife-edge case in Liu, Wang, and Wright (2011). Finally, both models without capital imply the economy collapses to autarky for high inflation rates.

We also remark that $\mathcal{W}^c > 0$ when A is close to $1 + r$. When A is large and hence the use of capital is permitted, the findings in Proposition 4 also have similar predictions as Proposition 3. First, under a sufficient condition ($v \leq \bar{v}$), the capital stock remains at its first-best level while the buyer's matching probability rises for a range of intermediate inflation rates. Second, capital overaccumulation is bound to occur for sufficiently high inflation rates. Together, our findings in Proposition 3 and Proposition 4 suggest that intermediate inflation affects the buyer's matching probability, but not capital accumulation. In addition, both models predict a Tobin effect but only for high inflation rates.

Figure 21: Output per Match

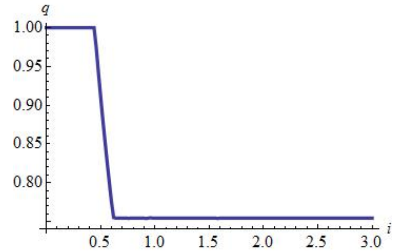


Figure 22: Measure of Buyers

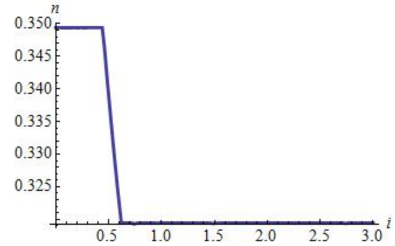


Figure 23: Aggregate Output

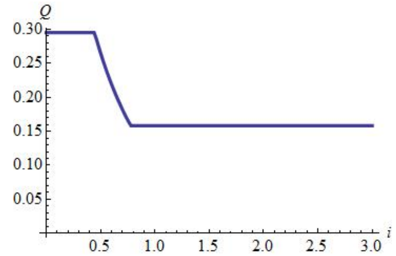


Figure 24: Matching Probability

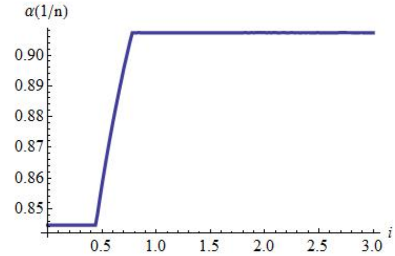


Figure 25: Capital

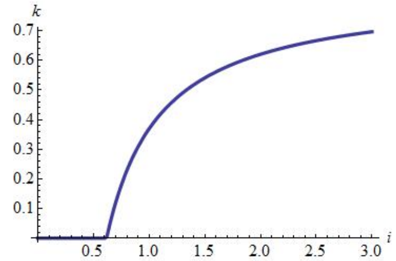
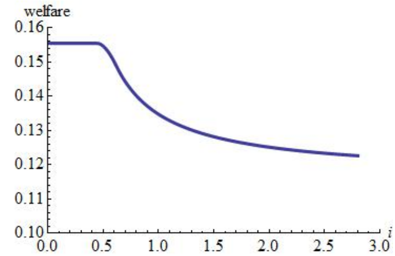


Figure 26: Welfare



Numerical examples illustrating the effects of inflation with endogenous participation are summarized in Figures 21–26. We set $b = 0.0001$, $\sigma = 0.5$, $\kappa = 1$, $r = 0.02$, $A = 0.9$, and $\upsilon = 0.4$. With $1 + r > A$, $k^* = 0$. According to Figure 24, the buyer's matching probability increases monotonically when inflation is in an intermediate range. In addition, the economy does not collapse into autarky, as can be seen in Figures 21 and 22 where output and the measure of buyers both remain strictly positive.

5 Concluding Remarks

In this paper, we adopt mechanism design to revisit some classical issues in monetary economics, namely the long run effects of inflation on output, search efforts, and capital accumulation as well as the social costs of inflation. We develop a tractable monetary model featuring costly search efforts to endogenize the frequency of trade, capital accumulation to endogenize the choice of a means of payment, and an endogenous trading mechanism that adjusts with the inflation tax.

The model is able to replicate several qualitative patterns emphasized in both empirical macro studies and historical anecdotes, including monetary superneutrality for a range of low inflation rates, non-linearities in trading frequencies and aggregate output, and substitution of money for capital for high inflation rates. While we acknowledge that certain aspects of our findings have appeared separately in previous studies, we show how they are intimately related by all being features of an optimal trading mechanism. That changes in inflation can have severe consequences on economic exchange and social interactions has also been emphasized by economic historians (Bresciani-Turroni (1931), Heynmann and Leijohnhufvud (1995), O'Dougherty (2002)). Nonetheless, there are important issues our paper abstracts from. Here we remark on a few caveats to our analysis and posit some directions for future research.

Optimal Trading Mechanism

In our framework, the economy's trading mechanism evolves to the optimal mechanism as the inflation rate changes. The inflation rate itself however is taken as exogenous, and our focus is to study the consequences of changes in inflation. Importantly however, we do endogenize society's trading mechanism and obtain very different results from previous studies, most of which treat the trading mechanism as a primitive. Indeed we show that under the optimal mechanism, the hot potato effect and substitution between money and capital are both optimal ways of responding to the inflation tax. Although it is unlikely for societies to change trading patterns for small changes in inflation, it seems plausible that societies would adjust trading mechanisms for large changes and our results are qualitatively in line with historical episodes of such changes.

We also remark that some of our results are robust to alternative trading mechanisms in the DM. Assuming agents meet in a centralized location in the DM, in the Supplemental Material, Section 2, we show that a version of our model with competitive pricing in the first stage also delivers non-monotone search intensity.[22] Due to the equivalence between competitive equilibrium and the core, this arrangement is still consistent with our mechanism design approach respecting the core requirement. However, there are some notable differences compared with our previous

[22]We thank Guillaume Rocheteau for making this suggestion to us.

findings. First, under competitive pricing, search intensity can rise with inflation but only near the Friedman rule. Since a strictly convex production cost delivers marginal cost pricing, the buyer's surplus can increase with inflation, but only for sufficiently small inflation rates. Second, the competitive equilibrium is generically inefficient due to the congestion externality: while the Friedman rule delivers the first-best level of output, search intensity is either too high or too low. Consequently, it is possible for inflation to increase welfare near the Friedman rule if search intensity is inefficiently low. Finally, we conjecture that it cannot deliver rate-of-return dominance for any inflation rate. While money and capital can coexist under Walrasian pricing, there will be rate-of-return equality, similar to Lagos and Rocheteau (2008).

Other Substitutes for Domestic Currency

Our model assumes that capital goods are the only alternative means of payments to money. However, capital goods in the model can be interpreted more broadly to include other real assets that may provide a hedge against inflation. This includes the use of assets not only for immediate settlement but as collateral (Caballero (2006)).[23] An example is the use of home equity as collateral to finance future consumption (Mian and Sufi (2011)). Moreover, individuals often resort to using foreign currencies for transactions during periods of high or hyperinflation (Calvo and Vegh (1992)). While our current framework cannot fully accommodate for the circulation of foreign currencies, an extension of our model to multiple countries and currencies is a fruitful topic for future research. Such a model could then determine how the presence of foreign currencies affects the consequences of inflation on international trade and welfare, as in Zhang (2014).

6 Appendix: Proofs

Proof of Proposition 1

We proved the necessity of constraints (12)-(16) in the main text. Here we prove their sufficiency. Let $(q^p, d_z^p, d_k^p, z^p, k^p, e^p)$ be an outcome that satisfies (12)-(16) and the pairwise core requirement. Consider the following trading mechanism with $R = F'(k^p)$:

[23] The role of assets as collateral also appears in Kiyotaki and Moore (2008) where assets do not change hands along the equilibrium path. This would entail DM trades using secured credit with capital playing the role of collateral. Then in the CM, debtors would settle obligations in numéraire. In our current set-up, capital goods are transferred between individuals and there is finality in each DM trade.

1. If $(z,k) \geq (z^p, k^p)$, then

$$
\begin{aligned}
o(z,k) \in \arg \max_{q,d_z,d_k} \{d_z + Rd_k - c(q)\} \\
\text{s.t. } \ u(q) - d_z - Rd_k \geq u(q^p) - d_z^p - Rd_k^p, \\
q \geq 0,\ d_z \in [0,z],\ d_k \in [0,k].
\end{aligned}
\tag{37}
$$

2. Otherwise,

$$
\begin{aligned}
o(z,k) \in \arg \max_{q,d_z,d_k} \{d_z + Rd_k - c(q)\} \\
\text{s.t. } \ u(q) - d_z - Rd_k \geq 0, \\
q \geq 0,\ d_z \in [0,z],\ d_k \in [0,k].
\end{aligned}
\tag{38}
$$

Solutions to the maximization problems (37) and (38) exist, and are denoted by $o(z,k) = [q(z,k), d_z(z,k), d_z(z,k)]$. Each solution has a unique $q(z,k)$. Although d_z and d_k may not be uniquely determined, we select the solution such that $d_z(z,k) = z$ if it exists and $d_k(z,k) = 0$ otherwise for any $(z,k) \neq (z^p, k^p)$. Indeed, in the Supplemental Material, Section 1, we show that the total wealth transfer is in fact uniquely determined.

To show that (q^p, z^p, k^p) is a solution to (37) for $(z,k) = (z^p, k^p)$, notice that (37) is the dual problem that defines the core of a pairwise meeting. Because $(q^p, z^p, k^p) \in \mathcal{CO}(z^p, k^p; R)$, it is also a solution to (37). This gives us a well-defined mechanism, o.

Now we show that the following strategy profile, (s_b^*, s_s^*), form a simple equilibrium: for all t and for all h^t, $(s_b^*)^{h^t,0}(z,k) = e^p$ if $(z,k) \geq (z^p, k^p)$, $(s_b^*)^{h^t,0}(z,k) = 0$ otherwise; for all portfolios (z,k), $(s_b^*)^{h^t,1}(z,k) = yes$ for all portfolios (z,k), and $(s_s^*)^{h^t,1}(z,k) = yes$; for all portfolios (z,k) and all responses (a_b, a_s), $(s_b^*)^{h^t,2}(z,k,a_b,a_s) = (z^p, k^p)$. In words, irrespective of their portfolios when entering the CM, buyers exit the CM with their proposed portfolios, (z^p, k^p). The effort choice is e^p if the buyer holds no less than the proposed portfolio in both assets; it is zero otherwise. In the DM they always say *yes* to the proposals. We show that s_b^* and s_s^* are optimal strategies following any history, given that all other agents follow (s_b^*, s_s^*).

Conditions (12) and (15), as well as the constraints in (37) and (38), ensure that both buyers and sellers are willing to respond with *yes* to the mechanism, both on and off equilibrium paths.

Now, by (37) and (38), the buyer's surplus is given by

$$
\begin{aligned}
&u\left[q(z,k)\right] - d_z\left(z,k\right) - Rd_k(z,k) = u(q^p) - d_z^p - Rd_k^p \text{ if } (z,k) \geq (z^p, k^p); \\
&u\left[q(z,k)\right] - d_z\left(z,k\right) - Rd_k(z,k) = 0 \text{ otherwise.}
\end{aligned}
\tag{39}
$$

As a result, because e^p satisfies (14) and $R = F'(k^p)$, it follows that $e(z,k) = e^p$ if $(z,k) \geq (z^p, k^p)$ and $e(z,k) = 0$ otherwise. Now consider the problem (7). By (39), any choice (z,k) with $(z,k) \geq (z^p, k^p)$ are strictly dominated by (z^p, k^p) and other choices are dominated by $(0,0)$, but (z^p, k^p) is better than $(0,0)$ by (12). This implies that (z^p, k^p) is the unique solution to the problem (7). □

Proof of Lemma 1

Under the constraint $k^p = 0$, by Proposition 1, a constrained-efficient outcome, (q^p, d_z^p, z^p, e^p), solves

$$\begin{aligned}
& \max_{(q,d_z,z,e)} e\alpha(1/e)[u(q) - c(q)] - \psi(e) && (40)\\
\text{subject to} \quad & -iz + e\alpha(1/e)[u(q) - d_z] - \psi(e) \geq 0, && (41)\\
& \psi'(e) = \alpha(1/e)[u(q) - d_z], && (42)\\
& -c(q) + d_z \geq 0, && (43)\\
& d_z \leq z, && (44)
\end{aligned}$$

and the pairwise core requirement.

Call the problem (40)-(44) program $\mathcal{A}$ and (21)-(24) program $\mathcal{B}$. Suppose (q^p, d_z^p, z^p, e^p) is a solution to $\mathcal{A}$. Because $z^p \geq d_z^p$,

$$-iz^p + e^p\alpha(1/e^p)[u(q^p) - d_z^p] - \psi(e^p) \leq -id_z^p + e^p\alpha(1/e^p)[u(q^p) - d_z^p] - \psi(e^p), \qquad (45)$$

and hence (q^p, d_z^p, e^p) satisfies the constraints in program $\mathcal{B}$. Now, if (q', d_z', e') has a higher value than that of (q^p, d_z^p, e^p), then, because (q', d_z', z', e') with $z' = d_z'$ also satisfies the constraints (pairwise core because $d_z' = z'$) in program $\mathcal{A}$, it also has a higher value than (q^p, d_z^p, z^p, e^p), a contradiction. Thus, (q^p, d_z^p, e^p) solves $\mathcal{B}$ as well.

Conversely, suppose that (q^p, d_z^p, e^p) solves program $\mathcal{B}$. Then, by setting $z^p = d_z^p$, (22) implies (41), and hence (q^p, d_z^p, z^p, e^p) satisfies the constraints in $\mathcal{A}$ (note that the pairwise-core is satisfied because $z^p = d_z^p$). If there is another (q', d_z', z', e') that gives a higher value than (q^p, d^p, z^p, e^p), then (q', d', e'), which also satisfies the constraints in $\mathcal{B}$, also gives a higher value than (q^p, d_z^p, e^p), a contradiction. Thus, (q^p, d_z^p, z^p, e^p) solves $\mathcal{A}$ as well.

We now show that any solution, (q^p, d_z^p, e^p), to (21)-(24) is such that $q^p \leq q^*$, $d_z^p \leq u(q^*)$, and $e^p \leq \hat{e}$ where $\hat{e}$ is given by (25).

To show that $q^p \leq q^*$, suppose by contradiction that $q^p > q^*$. Then the planner could decrease q^p to q^* to increase trade surplus while changing the transfer of real balances to $d_z' = u(q^*) - u(q^p) + d_z^p$.

Note that in the new outcome (q^*, d'_z, e^p) the buyer surplus remains the same, so constraints (22) and (23) hold. Because $q^* < q^p$, $u(q^*) - c(q^*) > u(q^p) - c(q^p) \geq u(q^p) - d^p_z$ and hence $d'_z \leq c(q^*)$, i.e., (24) holds. But the new outcome has a strictly higher welfare, a contradiction. This proves $q^p \leq q^*$. With $q^p \leq q^*$, it follows from (22) that $d^p_z \leq u(q^*)$. Then from (23), $\psi'(e^p)/\alpha(1/e^p) = u(q^p) - d^p_z \leq u(q^*) - c(q^*)$. Hence $e^p \leq \hat{e}$ such that $\hat{e}$ solves (25).

The above arguments also show that we may impose the constraints $q^p \leq q^*, d^p_z \leq u(q^*)$, and $e^p \leq \hat{e}$ with no loss in generality. Moreover with these additional constraints, (21)–(24) is a maximization problem with a continuous objective function and a compact feasible set. Hence a solution exists. □

Proof of Proposition 2

(1) We first show that for all $i \in [0, i^*]$, the first-best solution, (q^*, e^*, d^*_z), satisfies constraints (22)-(24). (23) holds by construction. Hence, $\alpha(1/e^*)[u(q^*) - d^*_z] = \psi'(e^*)$. Plugging this into (22), it is straightforward to verify that (22) holds if and only if

$$i \leq i^* = \frac{e^*\psi'(e^*) - \psi(e^*)}{u(q^*) - \psi'(e^*)/\alpha(1/e^*)}.$$

Finally, (24) holds if and only if $u(q^*) - \psi'(e^*)/\alpha(1/e^*) \geq c(q^*)$, that is, $\alpha(1/e^*)[u(q^*) - c(q^*)] \geq \psi'(e^*)$. But, by (20) and the fact that $\alpha'(\theta) > 0$ for all θ, $\alpha(1/e^*)[u(q^*) - c(q^*)] = [\alpha'(1/e^*)/e^*][u(q^*) - c(q^*)] + \psi'(e^*) > \psi'(e^*)$.

(2) We prove by two claims below.

Claim 1. (i) When $i = i^*$, the seller's constraint (24) holds with strict inequality at the optimum, (q^*, d^*_z, e^*); (ii) when $i > i^*$, the buyer's constraint (22) binds at the optimum; (iii) when $i > i^*$, $q^p < q^*$ at the optimum.

By Claim 1, for $i > i^*$, (22) binds and hence, together with (23), there is a unique solution for q and d_z as functions of e and i:

$$d_z(e, i) = \frac{e\psi'(e) - \psi(e)}{i}, \; q(e, i) = u^{-1}\left(\frac{e\psi'(e) - \psi(e)}{i} + \frac{\psi'(e)}{\alpha(1/e)}\right). \tag{46}$$

Let

$$g(e, i) = \frac{e\psi'(e) - \psi(e)}{i} + \frac{\psi'(e)}{\alpha(1/e)} \text{ and } f(x) = u^{-1}(x).$$

The objective function can be written as

$$\begin{aligned}G(e,i) &= e\alpha(1/e)\{u[q(e,i)]-c[q(e,i)]\}-\psi(e)\\ &= e\alpha(1/e)\{g(e,i)-c[f(g(e,i))]\}-\psi(e).\end{aligned}$$

Claim 2. There exists an $\bar{i}>i^*$ such that for all $i\in[i^*,\bar{i}]$, there is a unique maximizer $e^p(i)$ for $\max_{e\in[0,1]}G(e,i)$ and satisfies $\frac{d}{de}e^p(i)>0$ for all $i\in[i^*,\bar{i}]$. Moreover, $(q(e^p(i),i),e^p(i),d_z(e^p(i),i))$ is the unique constrained-efficient outcome for all $i\in[i^*,\bar{i}]$.

The results follow directly from Claim 2. Now we prove the two claims.

Proof of Claim 1. **(i)** We have shown it in (1).

(ii) To show that (22) binds for all $i>i^*$, consider the Lagrangian associated with (21), (22), (23), (24) (we ignore the nonnegativity constraints as they are irrelevant for this binding argument):

$$\begin{aligned}\mathcal{L}(q,d_z,e;\lambda,\mu,\eta) &= e\alpha(1/e)[u(q)-c(q)]-\psi(e)\\ &+ \lambda\{-id_z+e\alpha(1/e)[u(q)-d_z]-\psi(e)\}\\ &+ \mu\{d_z-c(q)\}\\ &+ \eta\{\psi'(e)-\alpha(1/e)[u(q)-d_z]\},\end{aligned}$$

where $\lambda\geq 0$, $\mu\geq 0$, η (which may be negative), are the Lagrange multipliers associated with (22), (24), and (23). From the Kuhn-Tucker Theorem, the following are the first-order necessary conditions taken with respect to q,d_z,e:

$$e^p\alpha(1/e^p)[u'(q^p)-c'(q^p)]+\lambda e^p\alpha(1/e^p)u'(q^p)-\mu c'(q^p)-\eta\alpha(1/e^p)u'(q^p)=0, \tag{47}$$

$$\begin{aligned}&[\alpha(1/e^p)-\alpha'(1/e^p)/e^p][u(q^p)-c(q^p)]-\psi'(e^p)+\lambda\{[\alpha(1/e^p)-\alpha'(1/e^p)/e^p][u(q^p)-d_z^p]-\psi'(e^p)\}\\ &=-\eta\{\psi''(e^p)+[\alpha'(1/e^p)/(e^p)^2][u(q^p)-d_z^p]\},\end{aligned} \tag{48}$$

$$\lambda[i+e^p\alpha(1/e^p)]=\mu+\eta\alpha(1/e^p). \tag{49}$$

In addition, (22) and (24) are not binding only if $\lambda=0$ and $\mu=0$, respectively.

Here we show that (22) binds at the optimum for all $i>i^*$. Suppose, by contradiction, that (22) does not bind and hence, $\lambda=0$. It also implies that $q^p>0$ and $e^p>0$. From (24) and $q^p>0$, we have $d_z^p>0$. Combining (47) and (49) yields

$$\frac{u'(q^p)}{c'(q^p)}=\frac{e^p\alpha(1/e^p)+\mu}{e^p\alpha(1/e^p)+\mu-\lambda i}. \tag{50}$$

From (50) and $\lambda=0$, $q^p=q^*$. From (49) and $\lambda=0$, $-\eta\alpha(1/e^p)=\mu$. Consider two cases. (a)

$\mu = 0$. Then, $\eta = 0$, and from (48), $e = e^*$, a contradiction. (b) $\mu > 0$. Then, (24) is binding and hence $d_z^p = c(q^*)$. By (49), $\mu = -\eta\alpha(1/e^p) > 0$. But then by (48),

$$[\alpha(1/e^p) - \alpha'(1/e^p)/e^p][u(q^*) - c(q^*)] - \psi'(e^p) > 0,$$

and hence $e^p < e^*$. However by (23),

$$\psi'(e^p) = \alpha(1/e^p)[u(q^*) - c(q^*)],$$

and, because $d_z^p = c(q^*) < d_z^*$, this implies $e^p > e^*$. This leads to a contradiction. This proves $\lambda > 0$ and hence (22) is binding.

(iii) Here we prove that $q^p < q^*$ for $i > i^*$. From (50), $q^p \neq q^*$ unless $\lambda = 0$, which is violated when $i > i^*$. Then $q^p < q^*$ follows from Lemma 1. □

Proof of Claim 2. We have three steps: first, we show that there is an open neighborhood $(e_0, e_1) \times (i_0, i_1)$ around (e^*, i^*) and a continuously differentiable implicit function $e^p : (i_0, i_1) \to (e_0, e_1)$ such that for all $i \in (i_0, i_1)$, $e^p(i)$ is the unique $e \in (e_0, e_1)$ such that $\frac{\partial}{\partial e} G(e^p(i), i) = 0$, and $e^p(i)$ is the unique local maximizer of $G(\cdot, i)$ in that neighborhood, with $(e^p)'(i) > 0$. Then, we show that, for some $i_2 \in (i^*, i_1]$ it is also the global maximizer for all $i \in (i^*, i_2]$. Finally, we show that we can obtain the constrained efficient outcome from $e^p(i)$.

First we show that

$$\frac{\partial G(e^*, i^*)}{\partial e} = 0, \ \frac{\partial^2}{\partial e^2} G(e^*, i^*) < 0 \tag{51}$$

as follows:

$$\begin{aligned}
\frac{\partial G(e^*, i^*)}{\partial e} &= \left[\alpha(1/e^*) - \frac{\alpha'(1/e^*)}{e^*}\right] [g(e^*, i^*) - c\,[f(g(e^*, i^*))]] \\
&+ e^*\alpha(1/e^*)\frac{\partial g(e^*, i^*)}{\partial e} \underbrace{[1 - c'[f(g(e^*, i^*))]f'(g(e^*, i^*))]}_{= 0 \text{ at } q = q^*} - \psi'(e^*) = 0. \\
\frac{\partial^2 G(e^*, i^*)}{\partial e^2} &= \left[\alpha(1/e^*) - \frac{\alpha'(1/e^*)}{e^*}\right] \frac{\partial g(e^*, i^*)}{\partial e} \underbrace{[1 - c'[f(g(e^*, i^*))]f'(g(e^*, i^*))]}_{= 0 \text{ if } q = q^*} \\
&+ \underbrace{[g(e^*, i^*) - c\,[f(g(e^*, i^*))]]}_{(+) \text{ since } u(q^*) - c(q^*) > 0} \underbrace{\frac{\alpha''(1/e^*)}{e^{*3}}}_{(-) \text{ since } \alpha'' < 0} - \underbrace{\psi''(e^*)}_{(+) \text{ since } \psi'' > 0} \\
&+ \underbrace{e\alpha(1/e^*) \left[\frac{\partial g(e^*, i^*)}{\partial e}\right]^2}_{(+)} \underbrace{[-c''[f'(g(e^*, i^*))]^2 - c'[f(g(e^*, i^*))f'']}_{(-)} < 0.
\end{aligned}$$

Then, by Implicit Function Theorem (IFT), there is an open neighborhood $(e_0, e_1) \times (i_0, i_1)$ around (e^*, i^*) and a continuously differentiable implicit function $e^p : (i_0, i_1) \to (e_0, e_1)$ such that for all $i \in (i_0, i_1)$, $e^p(i)$ is the unique $e \in (e_0, e_1)$ such that

$$\frac{\partial}{\partial e} G(e^p(i), i) = 0,$$

and that $\frac{\partial^2}{\partial e^2} G(e, i) < 0$ in that neighborhood (note that G is continuously twice differentiable). This shows that $G(e, i)$ is locally concave, and, as the unique solution to the first-order conditions, $e^p(i)$ is the local maximizer in that neighborhood.

To show that it is also the global maximizer, first consider $M(i) = \max_{e \notin (e_0, e_1)} G(e, i)$. By the Theorem of the Maximum, $M(i)$ is continuous and $M(i^*) < G(e^*, i^*)$. Let $\delta = G(e^*, i^*) - M(i^*) > 0$. Then by continuity, there exists an $i_2 \in (i^*, i_1]$ such that if $i \in [i^*, i_2]$, then $M(i) \le M(i^*) + \delta/3 < G(e^*, i^*) - \delta/3 \le G(e^p(i), i)$. Hence, for all $i \in [i^*, i_2]$, $e^p(i)$ maximizes $G(\cdot, i)$.

Because the function $-c(q(e, i)) + d_z(e, i)$ is continuous and because $-c(q^*) + d_z^* > 0$ by Claim 1 (i), it follows from continuity that there exists an $i_3 \in (i^*, i_2]$ such that for all $i \in (i^*, i_3]$, $-c(q(e^p(i), i)) + d_z(e^p(i), i) \ge 0$. Now we show that $(q(e^p(i), i), e^p(i), d_z(e^p(i), i))$ is the unique constrained-efficient outcome for $i \in (i^*, i_3]$. Suppose that (q', e', d'_z) solves (21)-(24). By Claim 1, (q', e', d'_z) satisfies (22) at equality and hence $q' = q(e', i)$ and $d'_z = d(e', i)$. It follows that $G(e', i) \le G(e^p(i), i)$. But, because (q', e', d'_z) is constrained efficient, $G(e', i) = G(e^p(i), i)$. Moreover, because we have a unique maximizer for $\max_e G(e, i)$, it follows that $e' = e^p(i)$ and hence $q' = q(e^p(i), i)$ and $d'_z = d_z(e^p(i), i)$. This proves that $(q^p, e^p, d_z^p) = (q(e^p(i), i), e^p(i), d_z(e^p(i), i))$ is the unique constrained-efficient outcome.

Finally, we show that $e^p(i)$ is strictly increasing in a neighborhood above i^*. By IFT again, $e^p(i)$ is continuously differentiable, and for all $i \in (i^*, i_3]$,

$$\frac{d}{di} e^p(i) = -\frac{\partial^2}{\partial e \partial i} G(e^p(i), i) / \frac{\partial^2}{\partial e^2} G(e^p(i), i). \tag{52}$$

We compute $\frac{\partial^2}{\partial e \partial i} G(e^*, i^*)$ as follows (note that $\{1 - c'[f(g(e^*, i^*))] f'(g(e^*, i^*))\} = 0$):

$$\begin{aligned} \frac{\partial^2}{\partial e \partial i} G(e^*, i^*) &= e^* \alpha (1/e^*) \underbrace{\frac{\partial}{\partial i} g(e^*, i^*)}_{(-)} \{ - \underbrace{c''[f(g(e^*, i^*))][f'(g(e^*, i^*))]^2}_{(+)} - \underbrace{c'[f(g(e^*, i^*))] f''}_{(+)} \} \underbrace{\frac{\partial}{\partial e} g(e^*, i^*)}_{(+)} \\ &> 0. \end{aligned}$$

By (51) and (52), $\frac{d}{di} e^p(i^*) > 0$. Because $e^p(i)$ is continuously differentiable, there exist an $\bar{i} \in (i^*, i_3]$

such that for all $i \in [i^*, \bar{i}]$, $\frac{d}{di} e^p(i) > 0$. To show that $d_z^p < d_z^*$ for $i \in (i^*, \bar{i}]$, we have from (23),

$$d_z^p = u(q^p) - \psi'(e^p)/\alpha(1/e^p) < u(q^*) - \psi'(e^*)/\alpha(1/e^*) = d_z^*,$$

since $q^p < q^*$ from Claim 1 (iii) and $e^p(i) > e^*$. $\square$

(3) By (23), $d_z^p = u(q^p) - \psi'(e^p)/\alpha(1/e^p)$. Thus, we may rewrite (22) and (24) as

$$\frac{\psi'(e^p)}{\alpha(1/e^p)} + \frac{e^p\psi'(e^p) - \psi(e^p)}{i} \geq u(q^p), \tag{53}$$

$$u(q^p) - c(q^p) \geq \frac{\psi'(e^p)}{\alpha(1/e^p)}. \tag{54}$$

By Lemma 1, for any constrained-efficient outcome, $e^p(i) \leq \hat{e} < 1$.

Fix some $e \in (0, \hat{e}]$. Let q_e satisfy

$$u(q_e) - c(q_e) = \psi'(e)/\alpha(1/e).$$

Since $\psi'(e)/\alpha(1/e)$ is increasing in e and continuous for $e \in (0, 1)$, it follows that $q_e \in (0, q^*]$ is uniquely determined and varies continuously in e. Let

$$i(e) = \frac{e\psi'(e) - \psi(e)}{u(q_e) - \psi'(e)/\alpha(1/e)}.$$

Then, $i(e) \in (0, \infty)$ and is continuous in e.

Now we show that if $i > i(e)$, then there is no q such that (e, q) satisfies (53) and (54) with respect to i. Suppose, by contradiction, that (e, q) satisfies (53) and (54) with respect to i. Then, by (54),

$$u(q) - c(q) \geq \psi'(e)/\alpha(1/e) = u(q_e) - c(q_e),$$

and hence, $q \geq q_e$. But by (53),

$$u(q) \leq \frac{\psi'(e)}{\alpha(1/e)} + \frac{e\psi'(e) - \psi(e)}{i} < \frac{\psi'(e)}{\alpha(1/e)} + \frac{e\psi'(e) - \psi(e)}{i(e)} = u(q_e),$$

which implies that $q < q_e$, a contradiction.

Finally, for each $e \in (0, \hat{e}]$, let

$$i_e = \max\{i(e') : e' \in [e, \hat{e}]\}.$$

Notice that i_e is well-defined because $i(e)$ is continuous and $[e, \hat{e}]$ is a compact set. Now, if $i > i_e$, then for any $e' \in [e, \hat{e}]$, $i > i_{e'}$ and hence (e', q) does not satisfy (53) and (54) with respect to i for any q. Thus, $e^p(i) < e$. $\square$

(4) By Claim 1 in (2), the buyer constraint (22) is binding for any constrained-efficient outcome, and thus we can determine d_z and q by (e, i). Recall that $d_z(e, i) = \frac{e\psi'(e) - \psi(e)}{i}$. Define $h(e) = \frac{\psi'(e)}{\alpha(1/e)}$. Then, $h' > 0$, $h(0) = 0$, $h(1) = \infty$, $\frac{\partial}{\partial e} d_z > 0$, $d_z(0, i) = 0$, and $d_z(1, i) = \infty$.

Recall also that $q(e, i) = u^{-1}[h(e) + d_z(e, i)]$ if (q, e) satisfies (22) at equality and (23). As in (3), we now work with (53) and (54). Then, (q, e) satisfies (53) if $q = q(e, i)$ and it satisfies (54) if, in addition to $q = q(e, i)$, $u(q) - c(q) \geq h(e)$. These conditions amount to $c^{-1} \circ u(d_z(e, i)) \geq h(e) + d_z(e, i)$. First we show that for e close to 1,

$$c^{-1} \circ u(d_z(e, i)) < h(e) + d_z(e, i). \tag{55}$$

By concavity of $c^{-1} \circ u$ and the Inada conditions, $\lim_{e \to 1} \frac{c^{-1} \circ u(d_z(e,i))}{d_z(e,i)} = 0$, and hence, for e sufficiently close to 1, $[c^{-1} \circ u(d_z(e, i))]/d_z(e, i) < 1 < 1 + h(e)/d_z(e, i)$, and this proves (55).

Here we show that for e sufficiently small,

$$c^{-1} \circ u(d_z(e, i)) > h(e) + d_z(e, i). \tag{56}$$

Notice that $\frac{\partial}{\partial e} d_z(e, i) = e\psi''(e)/i$ and $\psi''(0) = A > 0$ by assumption. Then,

$$\lim_{e \to 0} h'(e) = \lim_{e \to 0} \frac{\psi''(e)\alpha(1/e) + \psi'(e)\alpha'(1/e)/e^2}{\alpha(1/e)^2} \in [A, 2A],$$

where $\lim_{e \to 0} \alpha(1/e) = 1$, $\lim_{e \to 0} \alpha'(1/e)/e \leq \lim_{e \to 0} \alpha(1/e) = 1$, and $\lim_{e \to 0} \psi'(e)/e = \psi''(0) = A$. Note that the limit may not exist but for e small $h'(e)$ lies in the neighborhood of $[A, 2A]$. Thus, for sufficiently small e, $\frac{\partial}{\partial e} d_z(e, i) \in (eA/2i, 2Ae/i)$ and $h'(e) \in (A/2, 4A)$. For such e's, $d_z(e, i) + h(e) < (4A)e + (A/i)e^2$.

By assumption, there exists a $\delta < 0.5$ for which

$$\lim_{q \to 0} (c^{-1} \circ u)'(q) q^{0.5+\delta} > 0.$$

Therefore, for sufficiently small q, $(c^{-1} \circ u)'(q) > (0.5 - \delta) K q^{-0.5-\delta}$ for some $K > 0$, and hence $c^{-1} \circ u(q) > K q^{0.5-\delta}$ for all such q's. Thus, for e sufficiently small, $\frac{\partial}{\partial e} d_z(e, i) > eA/2i$ and hence $d_z(e, i) > e^2 A/i$, and we have

$$c^{-1} \circ u(d_z(e, i)) \geq K d_z(e, i)^{0.5-\delta} > K((A/4i)e^2)^{0.5-\delta} \equiv L e^{1-2\delta}.$$

Because $\lim_{e \to 0} \frac{Le^{1-2\delta}}{(4A)e+(A/i)e^2} = \infty$, it follows that, for e sufficiently small,

$$c^{-1} \circ u(d_z(e,i)) > Le^{1-2\delta} > (4A)e + (A/i)e^2 \geq h(e) + d_z(e,i).$$

This proves (56).

Now, by (55) and (56), and by the Intermediate Value Theorem, there exists $\tilde{e}_i > 0$ such that

$$d_z(\tilde{e}_i, i) = c \circ u^{-1}(h(\tilde{e}_i) + d_z(\tilde{e}_i, i)).$$

Then, $(q(\tilde{e}_i, i), \tilde{e}_i)$ satisfies (53) and (54). Moreover, the outcome $(q(\tilde{e}_i, i), \tilde{e}_i)$ is associated with positive welfare given by $\mathcal{W}(i)$ (we use (23) in the second equality):

$$\mathcal{W}(i) = \tilde{e}_i \alpha(1/\tilde{e}_i)[u(q(\tilde{e}_i, i)) - c(q(\tilde{e}_i, i))] - \psi(\tilde{e}_i) = \tilde{e}_i \psi'(\tilde{e}_i) - \psi(\tilde{e}_i) > 0.$$

□

Proof of Lemma 2

From Proposition 1, an outcome (q, d_k, k, e) is implementable if and only if

$$-[1 + r - F'(k)]k + e\alpha(1/e)[u(q) - F'(k)d_k] \geq \psi(e), \tag{57}$$

$$-c(q) + F'(k)d_k \geq 0, \tag{58}$$

$$1 + r \geq F'(k), \tag{59}$$

$$\psi'(e) = \alpha(1/e)[u(q) - F'(k)d_k], \tag{60}$$

and $(q, d_k) \in \mathcal{CO}(0, k; R)$ with $R = F'(k)$.

(1) Suppose that $(1+r)k^* \geq u(q^*) - \frac{\psi'(e^*)}{\alpha(1/e^*)}$. We show that the first-best allocation, (q^*, d_k^*, k^*, e^*), is implementable, where

$$d_k^* = \frac{1}{1+r}\left[u(q^*) - \frac{\psi'(e^*)}{\alpha(1/e^*)}\right].$$

Because $F'(k^*) = 1 + r$, (57) and (59) are satisfied. Note that $d_k^* \leq k^*$ because $(1 + r)k^* \geq u(q^*) - \frac{\psi'(e^*)}{\alpha(1/e^*)}$ and (60) is satisfied by construction. Finally, (58) holds if and only if $u(q^*) - \psi'(e^*)/\alpha(1/e^*) \geq c(q^*)$, that is, $\alpha(1/e^*)[u(q^*) - c(q^*)] \geq \psi'(e^*)$. But, by (20) and the fact that $\alpha'(\theta) > 0$ for all θ, $\alpha(1/e^*)[u(q^*) - c(q^*)] = [\alpha'(1/e^*)/e^*][u(q^*) - c(q^*)] + \psi'(e^*) > \psi'(e^*)$. □

(2) Suppose that $(1 + r)k^* < u(q^*) - \frac{\psi'(e^*)}{\alpha(1/e^*)}$. Here we show that $k^0 > k^*$ and hence the first-best is not implementable and that $\mathcal{W}^c > \mathcal{W}^0$.

First we show that $\mathcal{W}^0 > 0$. Consider the outcome $(\bar{q}, \bar{d}_k, k^*, \bar{e})$ given as follows: $\bar{q} = u^{-1}[(1+r)k^*] > 0$, $\bar{e}$ solves

$$[\alpha(1/e) - \alpha'(1/e)/e][u(\bar{q}) - c(\bar{q})] = \psi'(e),$$

$\bar{d}_k = u(\bar{q}) - \psi'(\bar{e})/\alpha(1/\bar{e}) > c(\bar{q})$. The outcome is implementable and is associated with positive welfare.

Second, we show that a constrained-efficient outcome (under the additional constraint $z = 0$), (q^c, d_k^c, k^c, e^c), exists. Note first that any outcome (q, d_k, k, e) with $q > q^*$ is strictly dominated by another outcome with $q' \leq q^*$; the proof follows exactly the same arguments as in the proof of Lemma 1. Second, any outcome (q, d_k, k, e) with $d_k < k$ is strictly dominated as well. If $k > k^*$, then we can decrease k and obtain higher welfare. Otherwise, assume that $k = k^*$ and consider two cases: (i) $q < q^*$. Then, consider another outcome (q', d_k', k, e) such that $q < q' < q^*$ and that $u(q') - F'(k)d_k' = u(q) - F'(k)d_k$. So buyer surplus is unchanged; the seller constraint is satisfied (note that $u(q') - c(q') > u(q) - c(q)$):

$$F'(k)d_k' - c(q') = u(q') - c(q') - u(q) + F'(k)d_k > -c(q) + F'(k)d_k \geq 0.$$

So (q', d_k', k, e) is implementable but has strictly higher welfare. (ii) $q = q^*$ and $k = k^*$. Then, because $(1+r)k^* < u(q^*) - \frac{\psi'(e^*)}{\alpha(1/e^*)}$ and because (q, d_k, k, e) satisfies (60), we have

$$\psi'(e)/\alpha(1/e) = [u(q^*) - (1+r)d_k] \geq [u(q^*) - (1+r)k^*] > \psi'(e^*)/\alpha(1/e^*), \tag{61}$$

and hence $e > e^*$. So lowering e will increase welfare. Consider (q, d_k', k, e') with $d_k < d_k' < k = k^*$ and that

$$\frac{\psi'(e')}{\alpha(1/e')} = [u(q^*) - F'(k^*)d_k'].$$

So $e' \in (e^*, e)$. Then, (q, d_k', k, e') is implementable but has strictly higher welfare.

Thus, we may only consider outcomes with $k = d_k$, and $q \leq q^*$. This implies that $k \leq \hat{k}$ that is given by

$$F'(\hat{k})\hat{k} = u(q^*). \tag{62}$$

Therefore, we may consider outcomes of the form (q, k, k, e) that satisfies (57)-(60) and $q \in [0, q^*]$, $k \in [0, \hat{k}]$. Thus, we have a maximization problem of a continuous objective function with a compact feasible set, which admits a maximum.

Now we show that, in any constrained-efficient outcome, (q^c, d_k^c, k^c, e^c), $k^c > k^*$. Suppose, by contradiction, that $k^c = k^*$. Consider two cases.

(a) $q^c < q^*$. We have shown that $d_k^c = k^*$. Note that because $k^c = k^*$ and because of (60), (57) holds with strict inequality. Let $k' > k^*$ be sufficiently close to k^* such that, by setting q' to satisfy $u(q') - F'(k')k' = u(q^c) - F'(k^*)k^*$, we have

$$q^c < q' < q^* \text{ and } e^c\alpha(1/e^c)[u'(q') - c'(q')]\frac{g'(k')}{u'(q^c)} > [1 + r - F'(k')],$$

where $g(k) = F'(k)k$, a concave function by assumption, and that (57) holds for $q = q'$, $e = e^c$, and $F'(k)d_k = F'(k')k'$. Note that the second requirement to define k' can be satisfied because the right-side is zero at k^* but the left-side is bounded away from zero. Because $u'(q') - c'(q') > u(q^c) - c(q^c)$, it follows that $F'(k')k' > c(q')$. Thus, (q', k', k', e^c) is implementable but the welfare difference is

$$e^c\alpha(1/e^c)[u(q') - c(q') - u(q^c) + c(q^c)] - \{[(1 + r)k' - F(k')] - [(1 + r)k^* - F(k^*)]\}$$

$$> e^c\alpha(1/e^c)[u'(q') - c'(q')]\frac{g'(k')}{u'(q^c)}[k' - k^*] - [(1 + r) - F'(k')][k' - k^*] > 0.$$

(b) $q^c = q^*$, and hence, by (61), $e^c > e^*$. Let $k' > k^*$ be sufficiently close to k^* such that, by setting e' to satisfy $u(q^*) - F'(k')k' = \psi'(e')/\alpha(1/e')$, we have

$$e^c > e' > e^* \text{ and } l'(e')\frac{g'(k')}{\max_{e\in[e',e^0]} j'(e)} > [1 + r - F'(k')],$$

where $j(e) = \psi'(e)/\alpha(1/e)$ and $l(e) = e\alpha(1/e)h(q^*) - \psi(e)$ (note that $j(e)$ is strictly increasing). Again, the second requirement that defines e' above can be satisfied because $1 + r - F'(k^*) = 0$ but the left-hand side is bounded away from zero. (q^*, k', k', e') is implementable but the welfare difference is

$$[l(e') - l(e^c)] - \{[(1 + r)k' - F(k')] - [(1 + r)k^* - F(k^*)]\}$$

$$> l'(e')\frac{g'(k')}{\max_{e\in[e',e^c]} j'(e)}[k' - k^*] - [(1 + r) - F'(k')][k' - k^*] > 0.$$

Therefore, we have $k^c > k^*$. □

Proof of Proposition 3

(1) The characterization and uniqueness of the first-best allocation as given by (18)-(20) follows similar arguments to those in Proposition 2 (1). Now we show that for all $i \in [0, i^{**}]$, the outcome $(q^*, d_z^*, k^*, z^*, k^*, e^*)$ with

$$d_z^* = u(q^*) - (1 + r)k^* - \frac{\psi'(e^*)}{\alpha(1/e^*)} > 0$$

satisfies constraints (12)-(16). Clearly, $F'(k^*) = 1+r$ implies (16) is satisfied. Note that (14) holds by construction. Plugging this into (12), it is straightforward to verify that it holds if and only if $i \leq i^{**}$ by definition of i^{**}. Note that (15) holds if and only if $u(q^*) - \psi'(e^*)/\alpha(1/e^*) \geq c(q^*)$, that is, $\alpha(1/e^*)[u(q^*) - c(q^*)] \geq \psi'(e^*)$. But, by (20) and the fact that $\alpha'(\theta) > 0$ for all θ,

$$\alpha(1/e^*)[u(q^*) - c(q^*)] = \alpha'(1/e^*)/e^*[u(q^*) - c(q^*)] + \psi'(e^*) > \psi'(e^*).$$

□

(2) First we show that when $i > i^{**}$, any outcome (q, d_z, d_k, z, k, e) with $d_z < z$ or $d_k < k$ is strictly dominated. Note that any outcome with $q > q^*$ is strictly dominated by another with $q' \leq q^*$. The case with $d_k < k$ follows the same arguments as those in the proof of Lemma 2. Consider the case with $d_z < z$ and $d_k = k$. If $k > k^*$, then we may decrease k and d_k and increase d_z to keep the buyer surplus unchanged, and by doing so we keep the constraints but increase the welfare. So assume that $k = k^*$. If $q < q^*$, then we may increase both q and d_z to keep the buyer surplus unchanged, and by doing so doing so we keep the constraints but increase the welfare. So assume that $k = k^*$ and $q = q^*$. Then, by (12) and (14),

$$\frac{e\psi'(e) - \psi(e)}{i} \geq z > d_z = u(q^*) - \psi'(e)/\alpha(1/e) - (1+r)k^*,$$

and hence

$$\frac{e\psi'(e) - \psi(e)}{u(q^*) - \psi'(e)/\alpha(1/e) - (1+r)k^*} > i > i^{**},$$

which implies that $e > e^*$. Thus, we may increase d_z and decrease e to keep (14) intact, and by doing so increase welfare.

Thus, we may only consider outcomes with $d_k = k$, $d_z = z$, and with $q \leq q^*$. Because $q \leq q^*$, to satisfy (12) it must be the case that $F'(k)k \leq u(q^*)$, that is, $k \leq \hat{k}$, which is given by (62). Thus, we may restrict attention to outcomes, (q, z, k, e), that satisfy

$$-iz - [1 + r - F'(k)]k + e\alpha(1/e)[u(q) - z - F'(k)k] \geq \psi(e), \tag{63}$$

$$-c(q) + z + F'(k)k \geq 0, \tag{64}$$

$$1 + r \geq F'(k), \tag{65}$$

$$\psi'(e) = \alpha(1/e)[u(q) - z - F'(k)k]. \tag{66}$$

Note that because $d_z = z$ and $d_k = k$, $(q, d_z, d_k) \in \mathcal{CO}(z, k; R)$.

Given these preliminary observations, we follow the same logic as the proof of Proposition 2, and prove the result by two claims.

Claim 1. (i) When $i = i^{**}$, the seller's participation constraint, (64), holds with strict inequality at the optimum, (q^*, z^*, k^*, e^*); (ii) for all $i > i^{**}$, the buyer's participation constraint, (63), binds, and $q^p < q^*$ at the optimum.

Given that (63) and (66) bind, we can solve for z and q as a function of (k, e, i):

$$z(k, e, i) = \frac{1}{i}\{e\psi'(e) - \psi(e) - [1 + r - F'(k)]k\},$$

$$q(k, e, i) = f\left\{g(e, i) + \frac{[-(1 + r) + (1 + i)F'(k)]k}{i}\right\},$$

where

$$g(e, i) = \frac{1}{i}[e\psi'(e) - \psi(e)] + \frac{\psi'(e)}{\alpha(1/e)} \text{ and } f(x) = u^{-1}(x).$$

The objective function can be written as

$$G(k, e, i) = e\alpha(1/e)\{u(q(k, e, i)) - c(q(k, e, i))\} - \psi(e) + F(k) - (1 + r)k. \tag{67}$$

Claim 2. There is an $\bar{i}'$ such that for all $i \in [i^{**}, \bar{i}']$, there is a unique maximizer, $(k^p(i), e^p(i))$, to

$$\max_{k \in [k^*, \hat{k}], e \in [0,1]} G(k, e, i),$$

with $z^p(i) = z[k^p(i), e^p(i), i] > 0$. Moreover, $(q^p(i), z^p(i), k^p(i), e^p(i))$ is the unique constrained-efficient outcome ($q^p(i) = q[k^p(i), e^p(i), i]$), and, if $1 + r + F''(k^*)k^* < \frac{-F''(k^*)k^*}{i^{**}}$, then $\frac{d}{de}e^p(i) > 0$ and $k^p(i) = k^*$.

The result follows directly from Claim 2. Now we prove the two claims.

Proof of Claim 1. **(i)** We have shown it in (1).

(ii) To show that (63) binds for all $i > i^{**}$, we consider two cases:

(a) At the optimum, $k^p > k^*$. Suppose, by contradiction, that (63) does not bind. Let (z', k') be such that $k^* \leq k' < k^p$ but $z' + F'(k')k' = z^p + F'(k^p)k^p$, and, by continuity, the tuple (q^p, z', k', e^p) also satisfies (63). Because $k' < k^p$, this leads to an increase in the welfare, a contradiction.

(b) At the optimum, $k^p = k^*$. Consider the Lagrangian associated with (63), (64), (65), (66),

$q \geq 0$, $z \geq 0$, and $e \geq 0$:

$$\begin{aligned}
\mathcal{L}(q,z,k,e;\lambda,\mu,\xi,\eta) &= e\alpha(1/e)[u(q)-c(q)] + [F(k)-(1+r)k] - \psi(e) \\
&+ \lambda\{-iz - [(1+r)-F'(k)]k + e\alpha(1/e)[u(q)-z-F'(k)k] - \psi(e)\} \\
&+ \mu\{[F'(k)k+z-c(q)]\} + \xi[(1+r)-F'(k)] \\
&+ \eta\{\psi'(e) - \alpha(1/e)[u(q)-z-F'(k)k]\},
\end{aligned}$$

where $\lambda \geq 0$, $\mu \geq 0$, $\xi \geq 0$, and η are the Lagrange multipliers associated with (63), (64), (65), and (66). From the Kuhn-Tucker Theorem, the following are the first-order necessary conditions with respect to q, z, e (with $k^p = k^*$):

$$e^p\alpha(1/e^p)[u'(q^p)-c'(q^p)] + \lambda e^p\alpha(1/e^p)u'(q^p) - \mu c'(q^p) - \eta\alpha(1/e^p)u'(q^p) = 0, \tag{68}$$

$$\lambda[i + e^p\alpha(1/e^p)] = \mu + \eta\alpha(1/e^p), \tag{69}$$

$$\begin{aligned}
&[\alpha(1/e^p) - \alpha'(1/e^p)/e^p][u(q^p)-c(q^p)] - \psi'(e^p) \\
&+\lambda\{[\alpha(1/e^p) - \alpha'(1/e^p)/e^p][u(q^p) - z^p - (1+r)k^*] - \psi'(e^p)\} \\
&= -\eta\{\psi''(e^p) + [\alpha'(1/e^p)/(e^p)^2][u(q^p) - z^p - (1+r)k^*]\},.
\end{aligned} \tag{70}$$

In addition, (63) and (64) are not binding only if $\lambda = 0$ and $\mu = 0$, respectively.

Here we show that (63) binds at the optimum for all $i > i^{**}$. Suppose, by contradiction, that (63) does not bind and hence $\lambda = 0$. It also implies that $q^p > 0$ and $e^p > 0$. Then from (64), $q^p > 0$, $e^p > 0$, and $k^p = k^*$, we have $z^p > 0$. Combining (68) and (69) yields

$$\frac{u'(q^p)}{c'(q^p)} = \frac{e^p\alpha(1/e^p) + \mu}{e^p\alpha(1/e^p) + \mu - \lambda i}. \tag{71}$$

From (71), $q^p = q^*$, and hence, from (69) and $\lambda = 0$ we have $-\alpha(1/e^p)\eta = \mu$. If $\mu = 0$, then from (70), $e^p = e^*$, a contradiction. If $\mu > 0$, then (64) is binding and hence $d_z^p + (1+r)k^* = c(q^*)$. By (69), $\mu = -\eta\alpha(1/e^p) > 0$. But then, by (70), this implies $[\alpha(1/e^p) - \alpha'(1/e^p)/e^p][u(q^*) - c(q^*)] - \psi'(e^p) > 0$, and hence $e^p < e^*$. However, by (66), $\psi'(e^p) = \alpha(1/e^p)[u(q^*) - c(q^*)]$, and, because $c(q^*) < d_z^* + (1+r)k^*$, this implies that $e^p > e^*$. This leads to a contradiction. Hence $\lambda > 0$ and so (63) is binding. Moreover, because $\lambda > 0$, (71) implies that $u'(q^p) > c'(q^p)$ and hence $q^p < q^*$. □

Proof of Claim 2. First note that

$$\begin{aligned}
\frac{\partial G(k^*,e^*,i^{**})}{\partial e} &= \left[\alpha(1/e^*)-\frac{\alpha'(1/e^*)}{e^*}\right][u(q^*)-c(q^*)]-\psi'(e^*)=0, \qquad (72)\\
\frac{\partial G(k^*,e^*,i^{**})}{\partial k} &= e^*\alpha(1/e^*)\left\{u'(q^*)-c'(q^*)\right\}\frac{\partial q(k^*,e^*,i^{**})}{\partial k}+F'(k^*)-(1+r)=0.
\end{aligned}$$

Now we show that

$$\frac{\partial^2 G(k^*,e^*,i^{**})}{\partial k^2}<0,\ \frac{\partial^2 G(k^*,e^*,i^{**})}{\partial e^2}<0,\ \frac{\partial^2 G(k^*,e^*,i^{**})}{\partial k^2}\frac{\partial^2 G(k^*,e^*,i^{**})}{\partial e^2}-\frac{\partial^2 G(k^*,e^*,i^{**})}{\partial k\partial e}>0. \qquad (73)$$

The second partial derivatives are

$$\begin{aligned}
&\frac{\partial^2 G(k^*,e^*,i^*)}{\partial e^2}\\
=\ &\alpha''(1/e^*)/(e^*)^3[u(q^*)-c(q^*)]+e^*\alpha(1/e^*)\left[\frac{\partial}{\partial e}q(k^*,e^*,i^{**})\right]^2[u''(q^*)-c''(q^*)]-\psi''(e^*)<0,\\
&\frac{\partial^2 G(k^*,e^*,i^*)}{\partial k^2}=e^*\alpha(1/e^*)\left[\frac{\partial}{\partial k}q(k^*,e^*,i^{**})\right]^2[u''(q^*)-c''(q^*)]+F''(k^*)<0,\\
&\frac{\partial^2 G(k^*,e^*,i^*)}{\partial k\partial e}=e^*\alpha(1/e^*)\left[\frac{\partial}{\partial k}q(k^*,e^*,i^{**})\frac{\partial}{\partial e}q(k^*,e^*,i^{**})\right][u''(q^*)-c''(q^*)].
\end{aligned}$$

Hence,

$$\begin{aligned}
&\frac{\partial^2 G(k^*,e^*,i^*)}{\partial e^2}\frac{\partial^2 G(k^*,e^*,i^*)}{\partial k^2}\\
>\ &e^*\alpha(1/e^*)\left[\frac{\partial}{\partial e}q(k^*,e^*,i^{**})\right]^2[u''(q^*)-c''(q^*)]e^*\alpha(1/e^*)\left[\frac{\partial}{\partial k}q(k^*,e^*,i^{**})\right]^2[u''(q^*)-c''(q^*)]\\
=\ &\left\{\frac{\partial^2 G(k^*,e^*,i^*)}{\partial k\partial e}\right\}^2.
\end{aligned}$$

Because of (72) and (73), and by the IFT, there is an open neighborhood $O=(k_0,k_1)\times(e_0,e_1)\times(i_0,i_1)$ around (e^*,i^*) and a continuously differentiable implicit function $(k_0^p,e_0^p):(i_0,i_1)\to(k_0,k_1)\times(e_0,e_1)$ such that for all $i\in[i^{**},i_1)$, $[k^p(i),e_0^p(i)]$ is the unique $(k,e)\in(k_0,k_1)\times(e_0,e_1)$ such that

$$\frac{\partial}{\partial e}G(k_0^p(i),e_0^p(i),i)=0 \text{ and } \frac{\partial}{\partial k}G(k^p(i),e_0^p(i),i)=0,$$

and another continuously differentiable implicit function $e^p:(i_0,i_1)\to(e_0,e_1)$ such that for all

$i \in [i^{**}, i_1)$, $e_1^p(i)$ is the unique $e \in (e_0, e_1)$ such that

$$\frac{\partial}{\partial e} G(k^*, e_1^p(i), i) = 0,$$

and that $G(\cdot, \cdot, i)$ is strictly concave over O. Now, define $(k^p(i), e^p(i))$ as

$$(k^p(i), e^p(i)) = \begin{cases} (k_0^p(i), e_0^p(i)) \text{ if } k_0^p(i) \geq k^* \\ (k^*, e_1^p(i)) \text{ otherwise.} \end{cases}$$

Because $G(\cdot, \cdot, i)$ is strictly concave over O, by the Kuhn-Tucker conditions, $(k^p(i), e^p(i))$ is a local maximizer; using the same arguments as those in Proposition 2, we can show that $(k^p(i), e^p(i))$ is the global maximizer as well, at least for some interval $[i^{**}, i_2]$ with $i_2 \in (i^{**}, i_1]$ and, using similar arguments there about seller participation constraint, one can show

$$(q^p, z^p, k^p, e^p) = (q[k^p(i), e_0^p(i), i], z[k^p(i), e_0^p(i), i], k^p(i), e_0^p(i))$$

is the unique constrained-efficient outcome for $i \in [i^{**}, i_2]$. Note that, by continuity, $e^p(i) > 0$ and $k^p(i)$ is close to k^* at least locally and hence $z^p > 0$.

Now we we show that if $1 + r + F''(k^*)k^* < \frac{-F''(k^*)k^*}{i^{**}}$, then $k^p(i) = k^*$ and $e^p(i)$ is increasing. For all $i \in [i^{**}, i_2]$, let $q(i) = q(k^*, e_1^p(i), i)$,

$$\frac{\partial}{\partial k} G(k^*, e_1^p(i), i) = e_1^p(i)\alpha(1/e_1^p(i))[u'(q(i)) - c'(q(i))]f'[u(q(i))] \left\{[1 + r + F''(k^*)k^*] + F''(k^*)k^*/i\right\}.$$

Because $1 + r + F''(k^*)k^* < \frac{-F''(k^*)k^*}{i^{**}}$, there exists $i_3 \leq i_2$ such that for all $i \in [i^{**}, i_3]$, $1 + r + F''(k^*)k^* + \frac{F''(k^*)k^*}{i} \leq 0$, and hence, for all such i's, $\frac{\partial}{\partial k} G(k^*, e_1^p(i), i) \leq 0$. Recall that $G(\cdot, \cdot, i)$ is strictly concave over O. Because

$$\frac{\partial}{\partial e} G(k^*, e_1^p(i), i) = 0 \text{ and } \frac{\partial}{\partial k} G(k^*, e_1^p(i), i) \leq 0$$

for all $i \in [i^{**}, i_2]$, it follows that $e^p(i) = e_1^p(i)$ for all $i \in [i^{**}, i_3]$ and hence the constrained efficient outcome has $k^p = k^*$.

Finally, by IFT again, $e^p(i)$ is continuously differentiable and for all $i \in (i^*, i_3]$,

$$(e^p)'(i) = -\frac{\partial^2}{\partial e \partial i} G(k^*, e^p(i), i) / \frac{\partial^2}{\partial e^2} G(k^*, e^p(i), i).$$

We have shown that $\frac{\partial^2}{\partial e^2}G(k^*, e^*, i^{**}) < 0$. Now,

$$\frac{\partial^2}{\partial e \partial i}G(k^*, e^*, i^{**}) = e^*\alpha(1/e^*)[u''(q^*) - c''(q^*)][f'(u(q^*))]^2 g_e(e^*, i^{**})g_i(e^*, i^{**}) > 0,$$

because

$$g_e(e^*, i^{**}) = \frac{e^*\psi''(e^*)}{i^{**}} + \frac{\psi''(e^*)\alpha(1/e^*) + \psi'(e^*)\alpha'(1/e^*)/(e^*)^2}{\alpha(1/e^*)^2} > 0, \; g_i(e^*, i^{**}) = \frac{e^*\psi'(e^*) - \psi(e^*)}{(i^{**})^2} < 0.$$

So $\frac{d}{di}e^p(i^{**}) > 0$ and, by continuity, there is $\bar{i}' \in (i^{**}, i_3]$ such that $\frac{d}{di}e^p(i) > 0$ for all $i \in [i^{**}, \bar{i}']$. $\square$

(3) Recall from Lemma 1 that for any i and in any constrained-efficient outcome w.r.t. i, $e^p(i) \le \hat{e}$. Note that the arguments there are not affected by the presence of capital. Moreover, by (63), we have

$$z^p(i) \le e^p(i)\alpha(1/e^p(i))[u(q^p(i)) - c(q^p(i))]/i \le \hat{e}\alpha(1/\hat{e})[u(q^*) - c(q^*)]/i.$$

Again, we prove the result by two claims below. Claim 3 show that $\mathcal{W}(i)$, the welfare associated with a constrained-efficient outcome under i, is arbitrarily close to $\mathcal{W}^k$ as i goes to infinity.

Claim 3. For any $\varepsilon > 0$, there exists i_ε for which $i > i_\varepsilon$ implies $\mathcal{W}(i) \le \mathcal{W}^k + \varepsilon$.

Because it is always feasible to set $z = 0$ and hence $\mathcal{W}(i) \ge \mathcal{W}^k$ for all i, the result that $\lim_{i\to\infty} \mathcal{W}(i) = \mathcal{W}^k$ follows immediately from Claim 3. By Lemma 2, if we impose the additional constraints $z = 0$ and $k = k^*$, then the resulting maximum welfare, denoted $\mathcal{W}^0$, is strictly less than $\mathcal{W}^k$, and hence $[\mathcal{W}^k - \mathcal{W}^0]/2 > 0$. The following claim shows that, if we impose $k = k^*$, then, for i sufficiently large, the maximum achievable welfare is less than $\mathcal{W}^k - [\mathcal{W}^k - \mathcal{W}^0]/2$. $\square$

Claim 4. Define $\mathcal{W}^0(i)$ to be the maximum welfare achievable by outcomes satisfying $k = k^*$, together with constraints (63)-(66). There exists an $\hat{i}$ such that for all $i > \hat{i}$, $\mathcal{W}^0(i) < \mathcal{W}^k - [\mathcal{W}^k - \mathcal{W}^0]/2$.

Claim 4 implies that for all $i > \hat{i}$, $k^p(i) > k^*$, for otherwise $\mathcal{W}(i) = \mathcal{W}^0(i) < \mathcal{W}^k$, a contradiction. Now we prove the two claims.

Proof of Claim 3. First note that in any constrained-efficient outcome, $q^p(i) \le q^*$ and $k^p(i) \le \hat{k}$. For each i, define $\tilde{k}(i)$ by the capital stock that satisfies

$$F'(\tilde{k}(i))\tilde{k}(i) - F'(\hat{k})\hat{k} = \frac{\hat{e}\alpha(1/\hat{e})[u(q^*) - c(q^*)]}{i}.$$

Because the function $F'(k)k$ is strictly increasing in k with range $\mathbb{R}_+$, $\tilde{k}(i)$ is well-defined and is a decreasing function of i. Moreover, as $i \to \infty$, $\tilde{k}(i)$ converges to $\hat{k}$.

Let $S(k) = F'(k)k$. Given $\epsilon > 0$, let i_ϵ be so large that $i > i_\epsilon$ implies

$$\{1 + r - F'[\tilde{k}(i)]\}[\tilde{k}(i) - \hat{k}] \leq \epsilon, \;\; S'(\tilde{k}(i))(1 + i) \geq 1 + r. \tag{74}$$

Note that i_ϵ is well-defined because $\tilde{k}(i)$ converges to $\hat{k}$ and S' is a decreasing function.

Now we show that if $i > i_\epsilon$, then $\mathcal{W}(i) \leq \mathcal{W}^k + \epsilon$. Fix some $i > i_\epsilon$, and a constrained-efficient outcome, $(q^p(i), d^p_z(i), d^p_k(i), z^p(i), k^p(i), e^p(i))$. Consider an alternative outcome

$$(q', d'_z, d'_k, z', k', e') = (q^p(i), 0, d'_k, 0, k', e^p(i)),$$

where k' and d'_k are such that

$$F'(k')k' - F'[k^p(i)]k^p(i) = z^p(i) \leq \frac{\hat{e}\alpha(1/\hat{e})[u(q^*) - c(q^*)]}{i}, \tag{75}$$

$$F'(k')d'_k = F'[k^p(i)]d^p_k(i) + d^p_z(i). \tag{76}$$

Note that $k' \leq \tilde{k}(i)$. Now we show that the outcome $(q', d'_z, d'_k, z', k', e')$ satisfies incentive compatibility constraints (63)-(66) and has welfare equal to $\mathcal{W}' \geq \mathcal{W}(i) - \epsilon$. Note that, by definition, $\mathcal{W}' \leq \mathcal{W}^k$ and hence this implies that $\mathcal{W}^k \geq \mathcal{W}(i) - \epsilon$.

First consider the buyer's participation constraint, (63). Because the original outcome satisfies (63), it suffices to show that

$$-iz^p(i) - [1 + r - F'(k^p(i))]k^p(i) \leq -[1 + r - F'(k')]k',$$

which holds if and only if

$$(1 + r)(k' - k^p(i)) - z^p(i) \leq iz^p(i) \Leftrightarrow (1 + r)(k' - k^p(i)) \leq (1 + i)z^p(i) \Leftrightarrow \frac{z^p(i)}{k' - k^p(i)} \geq \frac{1 + r}{1 + i}.$$

By definition of k', $z^p(i) = F'(k')k' - F'(k^p(i))k^p(i)$ and hence (note that $k' \leq \tilde{k}(i)$), by (74),

$$\frac{z^p(i)}{k' - k^p(i)} = \frac{F'(k')k' - F'(k^p(i))k^p(i)}{k' - k^p(i)} \geq S'(k') \geq S'(\tilde{k}(i)) \geq \frac{1 + r}{1 + i}.$$

In addition, because $k' \geq k^p(i)$ and because of (76), the alternative outcome satisfies (64)–(66).

Here we show that $\mathcal{W}' \geq \mathcal{W}(i) - \epsilon$. First note that

$$[F(k^p(i)) - (1 + r)k^p(i)] - [F(k') - (1 + r)k'] \leq [F'(k') - (1 + r)][k^p(i) - k'] = [1 + r - F'(k')][k' - k^p(i)].$$

Then, note that, in terms of variables relevant to the welfare, the alternative outcome differ from the original outcome only in the capital stock, and hence the difference in welfare, $\mathcal{W}' - \mathcal{W}(i)$, can be written as

$$\begin{aligned}\mathcal{W}' - \mathcal{W}(i) &= -\{[F(k^p(i)) - (1+r)k^p(i)] - [F'(k') - (1+r)k']\} \geq -[1+r-F'(k')][k' - k^p(i)] \\ &\geq -[1+r-F'(\tilde{k}(i))][\tilde{k}(i) - \hat{k}] \geq -\epsilon.\end{aligned}$$

The second last inequality follows from the fact that $k' - k^p(i) = z^p(i) \geq \tilde{k}(i) - \hat{k}$ and the fact that the function $S(k) = F'(k)k$ is concave in k, and the last inequality follows from (74). Hence, $\mathcal{W}' \geq \mathcal{W}(i) - \epsilon$. $\square$

Proof of Claim 4. We show that for any $\epsilon > 0$, there exists i'_ϵ such that $\mathcal{W}^0(i) < \mathcal{W}^0 + \epsilon$ for all $i > i'_\epsilon$. The claim follows immediately.

Because $\mathcal{W}^0(i) > 0$ (as it is always feasible to set $k = k^*$, q be such that $c(q) = (1+r)k^*$, and e that solves $\psi'(e)/\alpha(1/e) = [u(q) - (1+r)k^*] > 0$) for all i, we can find a lower bound $\underline{q}$ and $\underline{e}$ such that for any outcome $(q^0(i), d_z^0(i), d_k^0(i), z^0(i), k^0(i), e^0(i))$ that achieves the maximum welfare under the constraints (63)-(66) plus $k = k^*$, we have $q^0(i) > \underline{q}$ and $e^0(i) > \underline{e}$ for all i. Because $i > i^{**}$, at the optimum we must have $d_z^0(i) = z^0(i)$ and $d_k^0(i) = k^*$. Moreover, it follows that we can choose $u(\underline{q})$ to be strictly greater than $(1+r)k^*$, for otherwise the buyer will have arbitrarily small surplus and hence the search intensity will be arbitrarily small as well.

Now, the welfare, as a function of (q, k, e), is continuous and hence is uniformly continuous in $[\underline{q}, q^*] \times \{k^*\} \times [\underline{e}, \hat{e}]$. Thus, there exists $\delta > 0$ such that if $\|(q, e) - (q^0(i), e^0(i))\| < \delta$, then the welfare associated with (q, k^*, e), differs from the welfare $\mathcal{W}^0(i)$ by less than ϵ for all i.

Let $l(e) = \psi'(e)/\alpha(1/e)$. Then, $l'(e) > 0$ for all $e \in [\underline{e}, \hat{e}]$ and hence $A = \min_{e \in [\underline{e}, \hat{e}]} l'(e) > 0$. Let i'_ϵ be so large that if $i > i'_\epsilon$,

$$\max\{2, 1 + u'(\underline{q})/A\} \frac{[u(q^*) - c(q^*)]}{c'(\underline{q}/2)i} < \min\{\underline{q}/2, \delta/2, \underline{q} - u^{-1}[(1+r)k^*]\}, \tag{77}$$

Fix an $i > i'_\epsilon$ and an outcome $(q^0(i), z^0(i), k^*, e^0(i))$ that achieves $\mathcal{W}^0(i)$. We construct an alternative outcome, $(q', 0, k^*, e')$ such that $\|(q', e') - (q^0(i), e^0(i))\| < \delta$ and satisfies (63)-(66). Then, the welfare associated with the alternative outcome, denoted by $\mathcal{W}'$, is within ϵ of $\mathcal{W}^0(i)$, but $\mathcal{W}' \leq \mathcal{W}^0$.

The outcome $(q', 0, k^*, e')$ is given by

$$c(q') = c(q^0(i)) - z^0(i) \geq 0 \text{ and } \frac{\psi'(e')}{\alpha(1/e')} = u(q') - (1+r)k^*.$$

Because $z^0(i) \leq [u(q^*) - c(q^*)]/i$, it follows from (77) that $q' \geq \underline{q}/2$ and that $u(q') \geq (1+r)k^*$. Moreover, because $(q^0(i), z^0(i), k^*, e^0(i))$ satisfies (64),

$$-c(q') + (1+r)k^* = -c(q^0(i)) + z^0(i) + (1+r)k^* \geq 0,$$

and hence $(q', 0, k^*, e')$ satisfies (64) as well. Note that it also satisfies (63) and (66) by construction.

Thus, we have

$$0 \leq c(q^0(i)) - c(q') \leq z^0(i) \leq \frac{\hat{e}\alpha(1/\hat{e})[u(q^*) - c(q^*)]}{i},$$

and so, by (77),

$$|q^o(i) - q'| \leq |c(q^0(i)) - c(q')|/c'(\underline{q}/2) \leq \delta/2.$$

By (66),

$$|l(e^0(i)) - l(e')| = |\psi'(e^0(i))/\alpha(e^0(i)) - \psi'(e')/\alpha(1/e')| = |u(q^0(i)) - u(q') - z^0(i)| \leq u'(\underline{q}/2)[q^0(i) - q'] + z^0(i),$$

and so, by (77),

$$|e' - e^0(i)| \leq (1/A)[u'(\underline{q}/2)[q^0(i) - q'] + z^0(i)] < \delta/2.$$

Thus, we have $\|(q', e') - (q^0(i), e^0(i))\| < \delta$, and hence $\mathcal{W}^0 \geq \mathcal{W}' > \mathcal{W}^0(i) - \epsilon$. Finally, take $\hat{i} = i'_{[\mathcal{W}^k - \mathcal{W}^0]/2}$. $\Box$

Proof of Lemma 3

The necessity of those conditions are established in the main text. The sufficiency follows exactly the same arguments as those in the proof of Proposition 1. Note that as $n \to 1$, $\alpha(1/n) \to 0$, it follows that (32) cannot be satisfied at $n^p = 1$ with $v > 0$. $\Box$

Proof of Proposition 4

First we give a lemma.

Lemma 4. *Let $\bar{z}_i = \{[u(q^*) - c(q^*)] - v\}/i$ and let $\bar{k} = \{[u(q^*) - c(q^*)] - v\}/(1 + r - A)$. Then, There exists $(d_z^p, d_k^p) \leq (z^p, k^p)$ such that $(q^p, d_z^p, d_k^p, z^p, k^p, n^p)$ is a constrained-efficient outcome if* *the tuple* (q^p, z^p, k^p, n^p) *solves*

$$\max_{(q,z,k,n)\in[0,q^*]\times[0,\bar{z}_i]\times[0,\bar{k}]\times[0,n^*]} n\alpha(1/n)[u(q)-c(q)] - nv - n(1+r-A)k \tag{78}$$

subject to

$$-iz - (1+r-A)k + \alpha(1/n)[u(q)-z-Ak] - v = 0, \tag{79}$$

$$-c(q) + z + Ak \geq 0. \tag{80}$$

Moreover, if the first best allocation is not implementable, then for any constrained-efficient outcome, (q^p, z^p, k^p, n^p) *solves (78)–(80).*

Proof. For any solution, (q^p, k^p, n^p), that satisfies (78)-(80), the outcome

$$(q^p, d_z^p, d_k^p, z^p, k^p, n^p) = (q^p, z^p, k^p, z^p, k^p, n^p)$$

also satisfies (32)-(34). Here we show that, if the first-best is not implementable, then for any constrained-efficient outcome, $(q^p, d_z^p, d_k^p, z^p, k^p, n^p)$, $d_z^p = z^p$, $k^p = d_k^p$, and (q^p, z^p, k^p, n^p) satisfies (78)-(80) and belongs to the set $[0, q^*] \times [0, \bar{z}] \times [0, \bar{k}] \times [0, n^*]$. Notice that if $q^p > q^*$, then we may decrease q^p and increase z^p to keep the buyer's surplus and (32) unchanged but welfare is increased. So $q^p \leq q^*$. Moreover, if $n^p > n^*$, then, because $q^p \leq q^*$, we may decrease n^p and increase z^p to keep (32) unchanged but the welfare is increased. This also implies the pairwise core requirement is satisfied.

Now suppose that $(q^p, d_z^p, d_k^p, z^p, k^p, n^p)$ maximizes welfare subject to (32)-(34) and the pairwise-core. We show that (a) $k^p = d_k^p$ and (b) $z^p = d_z^p$.

(a) Suppose, by contradiction, that $k^p > d_p^k$. Then we may decrease k^p (and increase z^p proportionally to keep (32) unchanged) and increase $\mathcal{W}$, a contradiction.

(b) We consider two cases.

(b.1) Suppose that $q^p < q^*$ and $z^p > d_z^p$. Let $d_z' = d_z^p + \epsilon < z^p$ be such that $u(q') = u(q^p) + \epsilon \leq u(q^*)$. Then,

$$-iz^p - (1+r-A)k^p + \alpha(1/n^p)[u(q') - d_z' - Ad_k^p] = -iz^p - (1+r-A)k^p + \alpha(1/n^p)[u(q^p) - d_z^p - Ad_k^p] = v,$$

and

$$-c(q') + d_z' + Ad_k^p = -[c(q') - c(q^p) - \epsilon] + [-c(q^p) + d_z^p + Ad_k^p]$$

$$\geq \epsilon - c'(q')(q' - q^p) \geq \epsilon - u'(q')(q' - q^p) \geq \epsilon - [u(q') - u(q^p)] = 0.$$

Thus, $(q^p, d_z', d_k^p, z^p, k^p, n^p)$ is implementable but has higher welfare as $q' > q^p$, a contradiction.

(b.2) Suppose that $q^p = q^*$ and $z^p > d_z^p$. If $n^p < n^*$, then we can decrease z^p alone (without changing d_z^p) and increase n^p to keep (32) satisfied but increase the welfare, a contradiction. Suppose that $n^p = n^*$. Because the first-best is not implementable, $k^p = d_k^p > 0$. Then we may increase d_z^p and decrease k^p to make $d_z' + Ak' = d_z^p + Ak^p$ while changing z^p so that (32) is satisfied. Note that this is possible because $z^p > d_z^p$. Then, the welfare is increased, a contradiction. □

Proof of Proposition 4 proper.

First we consider a pure currency economy without capital. In that case, an outcome consists of (q^p, d_z^p, z^p, n^p). We have the following claim.

Claim 0. Consider an economy without capital, that is, with the additional restriction that $k = 0$. There exists (d_z^p, z^p) such that (q^p, d_z^p, z^p, n^p) is a constrained-efficient outcome if the pair (q^p, n^p) solves

$$\max_{(q,n)} n\alpha(1/n)[u(q) - c(q)] - nv \tag{81}$$

$$\text{subject to} \quad \alpha(1/n)[u(q) - c(q)] \geq ic(q) + v. \tag{82}$$

Proof. Suppose that (q^p, d_z^p, z^p, n^p) is a constrained-efficient outcome and suppose that (q^0, n^0) solves (81)-(82).

First we prove that (q^p, n^p) solves (81)-(82). Note that by implementability,

$$-iz^p + \alpha(1/n^p)[u(q^p) - d_z^p] = v, \ \ z^p \geq d_z^p, \ \ -c(q^p) + d_z^p \geq 0,$$

and hence

$$\alpha(1/n^p)u(q^p) - v \geq [i + \alpha(1/n^p)]d_z^p \geq [i + \alpha(1/n^p)]c(q^p).$$

This shows that $\alpha(1/n^p)[u(q^p) - c(q^p)] \geq v + ic(q^p)$, i.e., (q^p, n^p) satisfies (82). Now suppose that (q^0, n^0) gives a higher value than (q^p, n^p) to (81). Let

$$z^0 = d_z^0 = \frac{\alpha(1/n)u(q^0) - v}{i + \alpha(1/n)}.$$

Then, (q^0, d_z^0, z^0, n^0) is implementable (note that the pairwise core requirement is satisfied because $z^0 = d_p^0$), a contradiction to (q^p, d_z^p, z^p, n^p) being constrained efficient. So (q^p, n^p) solves (81)-(82).

Conversely, we show that (q^0, d_z^0, z^0, n^0) is constrained efficient. Suppose that it is not. Because it is implementable, it follows that (q^p, d_z^p, z^p, n^p) gives a higher value to (81) than (q^0, d_z^0, z^0, n^0). But (q^p, n^p) satisfies (82) and this leads to a contradiction to the fact that (q^0, n^0) solves (81). □

Now we turn to the proof of Proposition 4 proper.

(1) Let $i \in [0, i^*]$. Then, by Claim 0, to show that the first-best allocation, (q^*, n^*) and $k = 0$, is implementable, it is sufficient to show that (q^*, n^*) satisfies (82), that is,

$$i \leq \frac{\alpha(1/n^*)[u(q^*) - c(q^*)] - v}{c(q^*)} = i^*.$$

(2) Suppose that $i > i^*$. We prove the result by four claims. The first two claims consider the economy without capital, that is, with the additional constraint $k = 0$, and, by Claim 0, is concerned with the problem (81)-(82). Claim 1 shows that the constraint (82) binds at the optimum while Claim 2 shows that the optimal n^p decreases with i for a neighborhood of i^*. Claims 3 and 4 show that the constraint $k = 0$ is binding when $A > \bar{A}$ or $v > \bar{v}$ for a neighborhood of i^*.

Claim 1. Consider the economy without capital and suppose that $i > i^*$. Then, (82) binds at the optimum for the problem (81)-(82), and $q < q^*$ at the optimum.

The problem with (82) at equality simplifies to a choice of n: let $q = g(n, i) > 0$ solve

$$\alpha\left(\frac{1}{n}\right)[u(g(n,i)) - c(g(n,i))] = ic(g(n,i)) + v,$$

and then, substituting q by the function g, we may rewrite the objective function (81) as

$$n\alpha(1/n)\{u[g(n,i)] - c[g(n,i)]\} - nv = n \cdot i \cdot c[g(n,i)].$$

Thus, by Claim 1, the problem (81)-(82) can be reduced to

$$\max_{n \in [0, n^*]} n \cdot i \cdot c[g(n,i)]. \tag{83}$$

Claim 2. Consider the economy without capital and consider the corresponding maximization problem (83). There exists $\bar{i}' > i^*$ such that for each $i \in (i^*, \bar{i}']$, there exists a unique $n^p(i)$ that solves its associated F.O.C., and that is the global maximizer of the problem. Moreover, the outcome $(q^p(i), n^p(i)) = [g(n^p(i), i), n^p(i)]$ is the unique constrained-efficient outcome with $\frac{d}{di}n^p(i) < 0$.

Claim 2 shows that if we impose the constraint that $k = 0$, then for a range of inflation rates above i^*, there is a unique constrained-efficient outcome for each i in that range with the number of buyers entering the DM decreases with i. The next two claims show that the constraint $k = 0$ is binding in the economy with capital.

Claim 3. Consider the economy with capital and consider the problem (78)-(80). Suppose that $i > i^*$. Then, the constraint (80) binds at the optimum and $q^p < q^*$ at the optimum.

With (79) and (80) at equality, the problem becomes

$$
\begin{aligned}
&\max_{(q,z,k,n)} n\alpha(1/n)[u(q) - c(q)] - nv - n(1 + r - A)k \\
&\text{subject to} \\
&-iz - (1 + r - A)k + \alpha(1/n)[u(q) - c(q)] = v \\
&z + Ak = c(q).
\end{aligned}
$$

Claim 4. For any $i \in (i^*, i^0]$, where $i^0 = \frac{\alpha(1/n^*)[u(\overline{q})-c(\overline{q})]-v}{c(\overline{q})} > i^*$, $\overline{q} < q^*$ solves $\frac{u'(\overline{q})}{c'(\overline{q})} = 1 + \left(\frac{1+r-A}{A}\right)$, and for any constrained-efficient outcome, $(q^p(i), z^p(i), k^p(i), n^p(i))$, $k^p(i) = 0$.

By Claim 2 and Claim 4, and if we take $\overline{i} = \min\{i^0, \overline{i}'\}$, then for all $i \in (i^*, \overline{i}]$, there is a unique constrained-efficient outcome and $k^p(i) = 0$, $\frac{d}{di}n^p(i) < 0$. Now we prove the claims.

Proof of Claim 1. Consider the Lagrangian associated with the maximization problem (81) subject to (82):

$$
\begin{aligned}
\mathcal{L}(q, n; \lambda, \nu_q, \nu_n) &= n\alpha\,(1/n)\,[u(q) - c(q)] - nv \\
&+ \lambda\{-ic(q) + \alpha(1/n)[u(q) - c(q)] - v\},
\end{aligned}
$$

where $\lambda \geq 0$ is the Lagrange multiplier associated with (82). From the Kuhn-Tucker Theorem, the first-order necessary conditions with respect to q and n are

$$[\alpha(1/n^p)(n^p + \lambda)][u'(q^p) - c'(q^p)] - \lambda i c'(q^p) = 0 \tag{84}$$

$$\left[\alpha(1/n^p) - \alpha'(1/n^p)\left(\frac{1}{n^p} + \frac{\lambda}{(n^p)^2}\right)\right][u(q^p) - c(q^p)] = v. \tag{85}$$

To show that (82) binds for $i > i^*$; i.e. $\lambda > 0$, suppose by contradiction that $\lambda = 0$. Given $v > 0$ and (82) holds with strict inequality, $q^p > 0$ and $n^p > 0$. Hence, from (84) and (85), $q^p = q^*$ and $n^p = n^*$, a contradiction. Thus for $i > i^*$, $\lambda > 0$ and hence (80) binds.

To verify that $q^p < q^*$ for all $i > i^*$, first note that from (84), $q^p \neq q^*$ unless $\lambda = 0$, which is violated when $i > i^*$. Now suppose $q^p > q^*$ and consider a deviation that decreases q^p to q^*, which still satisfies (80). This produces higher welfare and is incentive feasible, a contradiction. Hence $q^p < q^*$. □

Proof of Claim 2. Fix some $i > i^*$. Define

$$f(n,i) \equiv i\{c[g(n,i)] + nc'[g(n,i)]g_n(n,i)\},$$

that is, $f(n,i) = \frac{\partial}{\partial n}\{n \cdot i \cdot c[g(n,i)]\}$. We apply the Implicit Function Theorem (IFT) in the neighborhood of the first-best to obtain a solution to $f(n,i) = 0$. To do so, we first determine the signs of the second derivatives, $f_n(n^*,i^*)$ and $f_i(n^*,i^*)$.

Let $h(q) = u(q) - c(q)$. Since $h'(q^*) = 0$, we have

$$\begin{aligned}
g_n(n^*,i^*) &= -\frac{c(q^*)}{n^*c'(q^*)} < 0;\ g_i(n^*,i^*) = -\frac{c(q^*)}{i^*c'(q^*)} < 0; \\
g_{ni}(n^*,i^*) &= \frac{c(q^*)}{i^*n^*c'(q^*)} + \frac{c(q^*)^2[\alpha(1/n^*)h''(q^*) - i^*c''(q^*)]}{(i^*)^2n^*[c'(q^*)]^3}; \\
g_{nn}(n^*,i^*) &= \frac{i^*\alpha''(1/n^*)h(q^*)c'(q^*) + \alpha'(1/n^*)h(q^*)\left[\frac{n^*c(q^*)}{c'(q^*)}[\alpha(1/n^*)h''(q^*) - i^*c''(q^*)] + 2n^*i^*c'(q^*)\right]}{(i^*)^2[c'(q^*)]^2(n^*)^4}.
\end{aligned}$$

Thus,

$$f(n^*,i^*) = i^*[c[g(n^*,i^*)] + n^*c'[g(n^*,i^*)]g_n(n^*,i^*)] = 0.$$

Moreover for $i^* > 0$, the second partial derivatives are

$$\begin{aligned}
f_n(n^*,i^*) &= i^*\left\{n^*c''[g(n^*,i^*)]g_n^2(n^*,i^*) + n^*c'[g(n^*,i^*))]g_{nn}(n^*,i^*) + 2c'[g(n^*,i^*)]g_n(n^*,i^*)\right\} \\
&= \underbrace{\frac{\alpha''(1/n^*)h(q^*)}{(n^*)^3}}_{(-)} + \underbrace{\frac{\alpha'(1/n^*)h(q^*)\left\{\frac{n^*c(q^*)}{c'(q^*)}[\alpha(1/n^*)h''(q^*)]\right\}}{i^*c'(q^*)(n^*)^3}}_{(-)} < 0.
\end{aligned}$$

$$\begin{aligned}
f_i(n^*,i^*) &= i^*\left\{c'[g(n^*,i^*)]g_i(n^*,i^*) + n^*c''[g(n^*,i^*)]g_n(n^*,i^*)g_i(n^*,i^*) + n^*c'[g(n^*,i^*)]g_{ni}(n^*,i^*)\right\} \\
&+ \underbrace{c[g(n^*,i^*)] + n^*c'[g(n^*,i^*)]g_n(n^*,i^*)}_{=0} \\
&= \underbrace{\frac{c(q^*)^2\alpha(1/n^*)h''(q^*)}{i^*[c'(q^*)]^2}}_{(-)} < 0.
\end{aligned}$$

Since $f(n^*,i^*) = 0$ and $f_n(n^*,i^*) < 0$, by the Implicit Function Theorem, there exists an open neighborhood $(n_0,n_1) \times (i_0,i_1)$ around the first-best (n^*,i^*) and a continuously differentiable function, $n^p : (i_0,i_1) \to (n_0,n_1)$ such that for $i \in (i^*,i_1)$, the function $n^p(i)$ gives the unique value of

$n \in (n_0, n_1)$ such that

$$f[n^p(i), i] = 0.$$

Because $f_n(n^*, i^*) = 0$ and because f is continuously differentiable, the objective function is locally concave and hence for some $i_2 \in (i^*, i_1]$, $n^p(i)$ is the local maximizer for the objective function. Following similar arguments as those used in Proposition 2 (2), we can also show that $n^p(i)$ achieves a maximum globally. Moreover, by the IFT again, for all $i \in (i^*, i_2]$,

$$\frac{d}{di} n^p(i) = -\frac{f_i[n^p(i), i]}{f_n[n^p(i), i]}.$$

Because $f_n(n^*, i^*) < 0$ and $f_i(n^*, i^*) < 0$, and because f is continuously differentiable, there exists $\bar{i}' \in (i^*, i_2]$ for which if $i \in (i^*, \bar{i}']$, then $f_n[n^p(i), i] < 0$ and $f_i[n^p(i), i] < 0$ and hence $\frac{d}{di} n^p(i) < 0$. Finally, $(q^p(i), n^p(i)) = (g(n^p(i), i), n^p(i))$ is a constrained-efficient outcome follows directly from Claim 0 and Claim 1. □

Proof of Claim 3. Consider the Lagrangian associated with (78)-(80), $z \geq 0$, and $k \geq 0$:

$$\begin{aligned}
\mathcal{L}(q, z, k, n; \lambda, \mu, \nu_z, \nu_k) &= n\alpha(1/n)[u(q) - c(q)] - nv - n(1 + r - A)k \\
&+ \lambda\{-iz - (1 + r - A)k + \alpha(1/n)[u(q) - z - Ak] - v\} \\
&+ \mu\{-c(q) + z + Ak\} \\
&+ \nu_z z + \nu_k k,
\end{aligned}$$

where λ, $\mu \geq 0$, $\nu_z \geq 0$, and $\nu_k \geq 0$ are the Lagrange multipliers associated with (79), (80), $z \geq 0$, and $k \geq 0$ respectively. The first-order necessary conditions with respect to q, z, k, n and the complementary slackness conditions for $z \geq 0$ and $k \geq 0$ are respectively

$$[\alpha(1/n^p)(n^p + \lambda)]u'(q^p) - [n^p\alpha(1/n^p) + \mu]c'(q^p) = 0 \tag{86}$$

$$\lambda[i + \alpha(1/n^p)] = \mu + \nu_z \tag{87}$$

$$-(1 + r - A)(n^p + \lambda) + A(\mu - \lambda\alpha(1/n^p)) + \nu_k = 0 \tag{88}$$

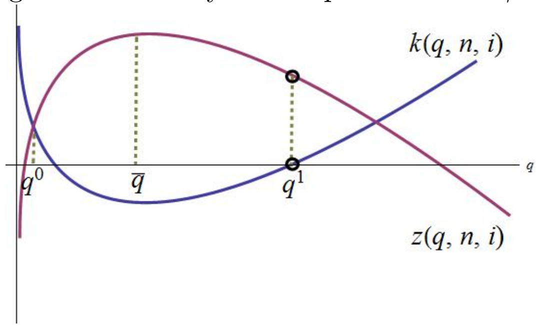
$$\begin{aligned}
&\left[\alpha(1/n^p) - \alpha'(1/n^p)/n^p\right][u(q^p) - c(q^p)] - (1 + r - A)k^p \\
&-\lambda[\alpha'(1/n^p)/(n^p)^2][u(q^p) - z^p - Ak^p] = v
\end{aligned} \tag{89}$$

$$\nu_z z^p = 0 \tag{90}$$

$$\nu_k k^p = 0. \tag{91}$$

(i) To show that (80) binds for all $i > i^*$, i.e. $\mu > 0$, we consider two cases.

Figure 27: Money and Capital When $A\gamma > 1$



(a) At the optimum, $k^p > 0$. Suppose by contradiction that (80) does not bind, i.e. $\mu = 0$. Consider (z', k') such that $0 \leq k' < k^p$ and $z' + Ak' = z^p + Ak^p$. By continuity, the allocation (q^p, z', k', n^p) still satisfies the seller's participation constraint, (80). But since $k' < k^p$, this leads to higher welfare, a contradiction.

(b) At the optimum, $k^p = 0$. Suppose by contradiction that (80) does not bind, i.e. $\mu = 0$. Then (80) holds with strict inequality and given $k^p = 0$, we have $z^p > 0$. Hence $\nu_z = 0$ and from (87), $\lambda = 0$. Hence $q^p = q^*$, $n^p = n^*$, and $k^p = 0$, which implies that the first-best is implementable for $i > i^*$, a contradiction. Hence (80) binds.

We now show that that $q^p < q^*$ for all $i > i^*$. First note that $q^p \neq q^*$ unless $\lambda = 0$ and $\nu_z = 0$ which is violated when $i > i^*$. Now suppose $q^p > q^*$ and consider a deviation such that q^p decreases to q^* and real balances are reduced to $z' = z^p - [u(q^p) - u(q^*)] \geq c(q^*)$. As $z' < z^p$, this deviation leads to higher welfare and is incentive feasible, a contradiction. Hence $q^p < q^*$ for $i > i^*$. □

Proof of Claim 4. (a) First consider the case where $A\gamma \leq 1$, or, equivalently, $A(i+1) \leq 1+r$. This means the rate of return on capital, $F'(k) = A$, is less than or equal to the rate of return on fiat money, γ^{-1}. Suppose that $k^p(i) > 0$. Then, let $k' = k^p(i) - \epsilon > 0$ and let $z' = z^p(i) + [(1+r-A)\epsilon]/i$. Then $(q^p(i), z', k', n^p(i))$ satisfies (79) and (80). Notice that $-iz^p(i) - (1+r-A)k^p(i) = -iz' - (1+r-A)k'$ by construction and

$$z' + Ak' = z^p(i) + Ak^p(i) + [1 + r - (1+i)A]\epsilon/i \geq z^p(i) + Ak^p(i)$$

because $1 + r - (1+i)A \geq 0$. Obviously the new outcome, $(q^p(i), z', k', n^p(i))$, is welfare-improving.
(b) Suppose now that $A\gamma > 1$ so that capital has a higher rate of return than fiat money. Given i and v and a choice of q and n, (79) and (80) at equality implies a unique solution for z and k given

by

$$
\begin{aligned}
z(q,n,i) &= \frac{-(1+r-A)c(q)+A\alpha(1/n)[u(q)-c(q)]-Av}{-(1+r)+A(1+i)} \\
&= \beta\left\{\frac{-(1+r-A)c(q)+A\alpha(1/n)[u(q)-c(q)]-Av}{A\gamma-1}\right\}, \\
k(q,n,i) &= \frac{-\alpha(1/n)[u(q)-c(q)]+ic(q)+v}{-(1+r)+A(1+i)} \\
&= \beta\left\{\frac{-\alpha(1/n)[u(q)-c(q)]+ic(q)+v}{A\gamma-1}\right\}.
\end{aligned}
$$

With $A\gamma > 1$ and $v > 0$, $z(q,n,i)$ is a concave function of q while $k(q,n,i)$ is a convex function of q as illustrated in Figure 27. Given i, using the fact that (79) holds at equality, the objective function simplifies to a choice of q and n:

$$\max_{q,n} n \cdot i \cdot z(q,n,i)$$

subject to $k(q,n,i) \geq 0$. Consider first the q^0 such that $k(q^0,n,i) = z(q^0,n,i)$. Then, $z(q,n,i) = c(q^0)/(A+1)$. While for q^1 such that $k(q^1,n,i) = 0$, $z(q,n,i) = c(q^1) > c(q^0)/(A+1)$ as $q^0 < \bar{q}$. Now we show that for all $n \in [0,n^*]$, and hence $\alpha(1/n) \in [\alpha(1/n^*),1]$, $z_q(q,n,i) > 0$ if $q > q^1$. Now,

$$z_q(q,n,i) = \beta\left\{\frac{-(1+r-A)c'(q)+A\alpha(1/n)[u'(q)-c'(q)]}{A\gamma-1}\right\},$$

and hence it suffices to show that $z_q(q^1,n,i) \leq 0$, that is,

$$-(1+r-A)c'(q^1)+A\alpha(1/n)[u'(q^1)-c'(q^1)] \leq 0.$$

It suffices to show that $q^1 \geq \bar{q}$, that is, $k(\bar{q},n,i) \geq 0$, as for all $n \in [0,n^*]$,

$$-(1+r-A)c'(\bar{q})+A\alpha(1/n)[u'(\bar{q})-c'(\bar{q})] \leq -(1+r-A)c'(\bar{q})+A[u'(\bar{q})-c'(\bar{q})] = 0.$$

Suppose by contradiction that $\overline{q} > q^1$. Then

$$-\alpha(1/n)[u(\bar{q})-c(\bar{q})]+ic(\bar{q})+v \leq -\alpha(1/n^*)[u(\bar{q})-c(\bar{q})]+ic(\bar{q}) \leq 0$$

as $i \leq i^0$. Notice that $i^0 > i^*$ because, as we have assumed,

$$v < \frac{\alpha(1/n^*)[c(q^*)(u(\bar{q})-c(\bar{q}))-c(\bar{q})(u(q^*)-c(q^*))]}{c(q^*)-c(\bar{q})}.$$

(3) Because $A = 0$, by Claim 0, a constrained-efficient outcome satisfies

$$\alpha(1/n)[u(q) - c(q)] \geq ic(q) + v.$$

For each $i \in \mathbb{R}_+$, let $\tilde{q}_i$ solves

$$u'(\tilde{q}_i) - c'(\tilde{q}_i) = ic'(\tilde{q}_i).$$

By concavity of u and convexity of c, one can verify that $\tilde{q}_i$ decreases with i and $\tilde{q}_i \to 0$ as $i \to \infty$. Then, for all $q \in \mathbb{R}_+$ and for all $n \in [0, 1]$,

$$\alpha(1/n)[u(q) - c(q)] - ic(q) \leq [u(\tilde{q}_i) - c(\tilde{q}_i)] - ic(\tilde{q}_i).$$

Let $\bar{i}$ be such that

$$[u(\tilde{q}_{\bar{i}}) - c(\tilde{q}_{\bar{i}})] = v.$$

Then, if $i > \bar{i}$, for all $n \in [0, 1]$ and for all $q \in \mathbb{R}_+$,

$$\alpha(1/n)[u(q) - c(q)] - ic(q) \leq [u(\tilde{q}_{\bar{i}}) - c(\tilde{q}_{\bar{i}})] - \bar{i}c(\tilde{q}_{\bar{i}}) < v.$$

So the only feasible allocation is autarky. □

(4) By (3), if $i > \bar{i}$ and if $k^p = 0$, then the outcome must be autarky. However, because $\mathcal{W}^c > 0$, there is an outcome with $k^p > 0$ with welfare $\mathcal{W}^c$. So we must have $k^p > 0$. □

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