Responding to the Inflation Tax*

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Abstract

This paper adopts a mechanism design approach to study the effects of anticipated inflation on trading patterns and welfare. We consider various channels through which individuals can respond to the inflation tax: search intensity (the intensive margin), market participation (the extensive margin), and substitution between money and capital. Instead of assuming a particular pricing protocol, we adopt a trading mechanism that maximizes society's welfare, taking as given the frictions in the economy. We show that inflation has non-monotonic effects on both the matching probability (the frequency of trades) and aggregate output (the total quantity of goods traded). For low inflation rates, money is superneutral. For moderate inflation rates, an increase in inflation induces a rise in the frequency of trades but has an ambiguous effect on aggregate output. However for high inflation rates, output eventually falls while search efforts may remain inefficiently high. In addition, individuals respond to high inflation by substituting money for capital as a means of payment. We show that non-monotone trading frequencies and the coexistence of money and capital are both features of an optimal mechanism. We also discuss how our results help rationalize the long-run effects of inflation documented in the literature.

Keywords: inflation, search intensity, money and capital, mechanism design

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1 Introduction

What are the effects of anticipated inflation on economic activity? This is a classical question in monetary economics and concerns not only how individuals respond to inflation but also the social costs of such responses. Insofar as inflation acts as a tax on cash transactions, the conventional wisdom is that individuals shift consumption away from cash-intensive activities as inflation rises. This aspect of inflation has been widely studied in the literature and is a key component determining the welfare cost of inflation (Cooley and Hansen (1989), Lucas (2000)). Similarly, the long-run relationship between inflation and real output has also received considerable attention in both theoretical and empirical studies.[1] While economists concede that the relationship between inflation and output is markedly different in low inflation environments than in high inflation environments, the consensus remains that except for low inflation rates, inflation is harmful to long-run output (King and Watson (1997), Ahmed and Rogers (2000), Bae and Ratti (2000)).

In addition to the aggregate effects of inflation on the purchasing power of money and real output, economists also highlight additional consequences of inflation that have received relatively less attention in the literature but are nonetheless critical to the functioning of monetary economies. These include (*i*) the effort taken by individuals engaging in market activities to economize on their money holdings, (*ii*) the accumulation of capital goods or other assets that may substitute for money as a means of payment, and (*iii*) the trade and exchange patterns that society adopts. While studies by economic historians reveal that their consequences for the functioning of monetary economies are both important and severe (Bresciani-Turroni (1931), Bernholz (2003)), we know of no existing study that can capture each of these aspects in a single coherent framework. Indeed there appears to be a disconnect between macroeconomists focusing on long-run aggregates and historians emphasizing micro-level trading behaviors, as the typical complete markets paradigm can say little about how inflation affects social interactions and society's exchange patterns.[2]

In this paper, we propose a unified framework that captures the consequences of inflation on aggregate activity and the three aspects of inflation highlighted above. In contrast with previous studies, our analysis pertains to low and moderate inflation economies as well as high and hyper-inflation economies. To capture the effects of inflation on individual money holdings and trade, we adopt a framework that has an explicit role for money, rather than as an argument in the util-

[1] Classical theoretical studies on the effects of inflation on real output include Tobin (1965), Sidrauski (1967), and Stockman (1981). Across these models however, there is no consensus on the effects of inflation on economic aggregates, due in part to the different ways in which money is modeled. It is thus desirable to derive such a hypothesis using a microfounded model where money's social role arises endogenously.

[2] Exceptions include studies by Casella and Feinstein (1990) and Tommasi (1994, 1999) on economic exchange during hyperinflation. However, since these papers assume that domestic money is required for all transactions, individuals cannot respond to high inflation by substituting to other assets that are not subject to the inflation tax.

ity function (Sidrauski (1967)) or as a constraint for transactions (Stockman (1981), Cooley and Hansen (1989)). Our baseline model features alternating rounds of centralized trades and pairwise meetings where a double-coincidence problem and frictions such as limited commitment, lack of enforcement, and lack of record-keeping make money essential for trade. As in Lagos and Wright (2005), quasi-linear preferences make the model analytically tractable albeit at a cost. While shutting down the distributional effects of inflation, the advantage of tractability allows us to derive new insights by characterizing analytically the consequences of inflation mentioned earlier.[3] As we show, this departure from previous studies matters significantly for capturing the qualitative relationships between inflation and economic aggregates that have appeared historically.

To emphasize the channels through which inflation affects economic activity and trade, our framework modifies the Lagos and Wright (2005) model along three critical dimensions. First, the matching probability and hence the frequency of trade is determined endogenously by buyers' costly search efforts. Indeed one of the most salient features of high inflations is that individuals try to speed up their purchases in order to reduce the time they carry money for transactions. As Irving Fisher put it, "when depreciation is anticipated, there is a tendency among owners of money to spend it speedily" (Fisher (1911)). This behavior appears in descriptive accounts of life experiences during the 1920s hyperinflation in Germany by Bresciani-Turroni (1931), the chronic high inflations of Argentina and Brazil in the 1990s by Heynmann and Leijohnhufvud (1995) and O'Dougherty (2002), and more recently in Zimbabwe in 2006.[4] These historical narratives describe the so-called "hot potato" effect of inflation where individuals expend valuable time and effort trying to spend their money more quickly that would be spent more efficiently without high inflation. As in Lagos and Rocheteau (2005), we endogenize search effort as a natural way to study this hot potato effect, a phenomenon that has been elusive to capture in previous studies.

Second, we allow individuals to accumulate capital goods that can compete with money as a means of payment. In practice, societies in times of high inflation tend to use other assets for transaction purposes, including real assets or other currencies that do not depreciate with domestic inflation (Calvo and Vegh (1992), Porter and Judson (1996)). A related observation is that rising inflation induces a Tobin-like effect resulting in an increase in capital accumulation.[5]

[3] See Wallace (2014) for additional discussion on the distributional effects of inflation.

[4] Anecdotal evidence on the distortionary effects of the 1920s German hyperinflation indicate workers being paid multiple times per day and shopping midday to avoid further depreciation of their earnings (Bresciani-Turroni (1931)). Similarly, Guttmann and Meehan (1975) describe the experiences of individuals dealing with high inflation in Germany from 1919 to 1923 where on average, prices quadrupled each month during the sixteen months of hyperinflation: "At eleven o'clock in the morning a siren sounded and everybody gathered in the factory forecourt where a five-ton lorry was drawn up loaded brimful with paper money. The chief cashier and his assistants climbed up on top. They read out names and just threw out bundles of notes. As soon as you had caught one you made a dash for the nearest shop and bought just anything that was going."

[5] Bresciani-Turroni (1931) documents the effects of capital overaccumulation in Weimar Germany from 1914–1923:

For instance, Bernholz (2003) notes that during periods of high inflation, "[f]actors of production are moved from producing consumer goods to that of investment goods[...] As a consequence, high and erratic inflation has a devastating effect on the inter-temporally efficient allocation of resources." Introducing capital goods with a potential transactions role allows us to study the extent to which inflation induces individuals to substitute between money and another asset with a better store of value, and relatedly, how inflation affects capital accumulation.[6] This contrasts sharply with many previous studies of inflation which typically assume a cash-in-advance constraint that restricts individuals from using other assets to better cope with the inflation tax.

Third, we adopt a mechanism design approach where the economy's trading mechanism is determined endogenously to maximize social welfare for any given inflation rate. As in Hu, Kennan, and Wallace (2009), the trading mechanism dictates how to divide the gains from trade between a buyer and seller, conditional on the pair's composition of asset holdings. This approach allows us to consider how inflation affects the trading arrangements that can be implemented by society and hence the economy's market structure. As emphasized by Casella and Feinstein (1990), inflation not only affects how individuals economize on their real balances, it also changes the economy's trading patterns: "[H]istorians emphasize hyperinflation's disruptive impact on individuals and on their socioeconomic relationships. Previously stable trading connections were severed, transactions patterns were altered, and normally well-functioning markets collapsed." Moreover, as pointed out by Hu and Rocheteau (2013), the optimal trading mechanism features rate-of-return-dominance and the coexistence of money and capital, while such coexistence does not occur under suboptimal trading mechanisms.

Our results provide a comprehensive picture on the consequences of inflation for a range of inflation rates. As a benchmark, we first consider an economy with money only and hence no production of capital. For a range of low inflation rates, money is superneutral: output and search efforts remain at their first-best levels, irrespective of changes in inflation. When inflation rises above a certain threshold, both the buyer's surplus and search effort increase with inflation, even though the buyer's holdings of real balances falls. Hence for moderate inflation rates, our model exhibits two countervailing effects: in the decentralized market (DM), an increase in inflation leads to a higher frequency of trade but lower output per trade. The total quantity of goods traded in the DM can therefore increase with inflation if the former effect dominates the latter. Indeed

"To avoid the effects of the monetary depreciation, German agriculturists continued to buy machines; the 'flight from the mark to the machine,'... was the most convenient and the easiest means of defence against the depreciation of the currency. But towards the end of the inflation, farmers realized that a great part of their capital was sunk in machines, whose number was far above what would ever be needed."

[6] Orphanides and Solow (1990) comment on the desirability of incorporating capital accumulation in environments with high inflation: "We know of no study of hyperinflations that mentions the Tobin effect! If the money and growth literature is relevant to anything this must be where it fits in."

we present numerical examples where moderate inflation actually induces DM aggregate output to rise, a finding consistent with the non-linear relationship between inflation and output found in Bullard and Keating (1995).[7] In addition, since search effort is above its first-best level, inflation has an additional welfare cost, apart from its effect on real output. Without the presence of capital, however, search efforts eventually fall towards zero as inflation tends to infinity. This result should not be surprising: as extreme rates of inflation brings monetary trade to near collapse, buyers lose any incentive to search. However, that most economies do not turn into complete barter societies even under hyperinflation suggests that other assets may be used for transactions, a possibility we turn to next.

We highlight another channel for individuals to deal with inflation by introducing capital goods that can compete with money as a means of payment. Following Lagos and Rocheteau (2008), capital goods produced in the centralized market (CM) can be used to produce consumption goods in the next period's CM. For low inflation rates, monetary superneutrality still holds: the capital stock, together with other welfare-relevant variables, remains at its first-best level irrespective of changes in inflation. Moreover, under some conditions, moderate inflation induces search efforts to increase while the capital stock remains at its first-best level. However, overaccumulation of capital is bound to occur as inflation rises, even though search efforts may remain inefficiently high. This result contrasts dramatically with what happens without capital, where the economy is guaranteed to approach autarky under hyperinflation. Our findings also suggest that capital overaccumulation is a symptom only of high inflation rates, while Tobin effects are small or absent for moderate inflation rates.

Finally, we relate our findings to some previous results in the literature. A key aspect of our findings is that the hot potato effect and the coexistence of money and capital are both features of the optimal mechanism. The effect of inflation on search efforts has been addressed previously in search theoretic models, namely by Li (1994, 1995, 1997), Lagos and Rocheteau (2005), Ennis (2008, 2009), Nosal (2011), Liu, Wang, and Wright (2011), and Dong and Jiang (2014), among others.[8]

[7] Bullard and Keating (1995) find that a permanent increase in inflation has a positive effect on output at low inflation rates that dissipate for higher levels. Rapach (2003) confirms this positive correlation in a sample of 14 OECD countries using a structural autoregression framework, while Bae and Ratti (2005) document a negative correlation for high inflation countries.

[8] There are alternative approaches to generate the so-called hot potato effect. Under competitive search, Lagos and Rocheteau (2005) show that low inflation can increase the buyer's surplus even though total surplus decreases, but only for certain parameters and only for inflation rates close to the Friedman rule. Another way to generate the hot potato effect is to have periodic access to the centralized market, as in Ennis (2009), or to introduce preference shocks, as in Peterson and Shi (2004), Faig and Jerez (2006), Ennis (2008), Nosal (2011), and Dong and Jiang (2014). Liu, Wang, and Wright (2011) focus instead on buyers' participation decisions and show that inflation decreases the number of buyers, thereby increasing the frequency of trades. In Section 4.3, we show that a similar finding arises in our setting under an optimal trading mechanism.

However, the findings from this literature are mixed and suggests that whether inflation induces search efforts to rise or fall is sensitive to the assumed market structure or pricing mechanism, which may introduce its own inefficiencies. In contrast, our finding that search efforts can increase with inflation arises as a feature of the optimal mechanism. While our results regarding non-monotone search intensity may appear similar to Lagos and Rocheteau (2005) under competitive search, the environments are very different and hence a direct comparison is misleading. Nonetheless, there are several important differences in our results that we discuss further in Section 4.1. Regarding the coexistence of money and higher return assets, Hu and Rocheteau (2013) show that rate-of-return dominance and the coexistence of money and capital are features of good allocations. Our results extend their findings to a model with endogenous search efforts and we find a non-trivial interaction between the two: in the presence of capital, search efforts may remain high as inflation increases while without capital, search efforts eventually fall.

This paper proceeds as follows. Section 2 describes the baseline environment with endogenous search intensity and the possibility for money and capital to compete as means of payment. Section 3 describes implementation of the optimal mechanism, and Section 4 characterizes the effects of inflation on output, search effort, capital accumulation, and welfare. We also consider an alternative formalization where buyers can choose to participate in market activities in lieu of search intensity and analyze the effects of inflation along the extensive margin. Finally, Section 5 concludes.

2 Environment

Time is discrete and has an infinite horizon. The economy is populated by a continuum of infinitely-lived agents, divided into a set of *buyers*, denoted by $\mathbb{B}$, and a set of *sellers*, denoted by $\mathbb{S}$. Each date has two stages: the first has pairwise meetings in a decentralized market and the second has centralized meetings. The first stage will be referred to as the *DM* (decentralized market) while the second stage will be referred to as the *CM* (centralized market). Time starts in the CM of period 0.

There is a single perishable good at each stage, with the CM good taken as the numéraire. In the CM, all agents have the ability to produce and wish to consume. Agents' labels as buyers and sellers depend on their roles in the DM where only sellers are able to produce and only buyers wish to consume. All agents have a constant discount factor $\beta = \frac{1}{1+r} \in (0, 1)$.

The numéraire good can be transformed into a capital good one for one. Capital goods accumulated at the end of period t are used by sellers at the beginning of the CM of $t + 1$ to produce the numéraire good according to the technology $F(k)$, where F is twice continuously differentiable,

strictly increasing, strictly concave, and satisfies the Inada conditions $F'(0) = \infty$ and $F'(\infty) = 0$.[9] We also assume that the function $F'(k)k \in \mathbb{R}_+$ is strictly increasing and strictly concave in k. Capital goods depreciate fully after one period, and the rental (or purchase) price of capital in terms of the numéraire good at period t is denoted R_t.[10] The assumption of full depreciation is with no loss in generality. For instance, we could have assumed a production technology $f(k)$ and depreciation rate $\delta \in (0, 1)$, and then define $F(k)$ as $F(k) = f(k) + (1 - \delta)k$, which will give us exactly the same analytical results.

There is also an intrinsically useless, perfectly divisible and storable asset called money. Let M_t denote the quantity of money in the CM of period t. The relative price of money in terms of the numéraire is denoted ϕ_t. There is an exogenously given gross growth rate of the money supply, which is constant over time and equal to γ; that is, $M_{t+1} = \gamma M_t$.[11] New money is injected if $\gamma > 1$, or withdrawn if $\gamma < 1$, by lump-sum transfers or taxes, respectively.[12] Transfers take place at the beginning of the CM and we specify that they go to buyers only. Lack of record-keeping and private information about individual trading histories rule out unsecured credit, giving a role for money and capital to serve as means of payment. In addition, individual money holdings are common knowledge in a match. We assume that sellers do not carry real balances or capital across periods. As shown in Hu and Rocheteau (2013), this assumption is with no loss of generality.

Agents are matched pairwise and at random in the DM. We normalize the measure of sellers and buyers each to one. We assume that the seller's search intensity is exogenously given. However, buyers can choose their search intensity at the beginning of each period. At the beginning of the DM, each buyer $b \in \mathbb{B}$ chooses search intensity, $e_b \in [0, 1]$. The average search intensity of buyers is $\overline{e}$, defined as

$$\overline{e} = \int_{b \in \mathbb{B}} e_b db.$$

A buyer exerting effort e to search in the DM incurs a cost in utility terms of $\psi(e)$. We assume that for all $e \in [0, 1)$, $\psi(e) \in [0, \infty)$ is twice continuously differentiable, strictly increasing, strictly convex, and satisfies the Inada conditions $\psi(0) = \psi'(0) = 0$, $\lim_{e \to 1} \psi(e) = \infty$, and $\lim_{e \to 1} \psi'(e) =$

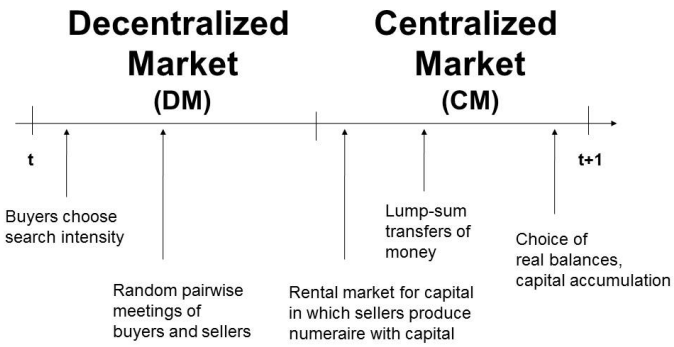
[9] In other words, we assume "putty-putty" capital as in Romer (1990), where capital can be transformed into the numéraire good and then back again to capital.

[10] Because there is full depreciation of capital each period, it is equivalent for agents to buy or rent capital goods.

[11] As our focus is to study trading patterns across different inflation rates, the money growth rate is not chosen optimally and is taken as given in the mechanism design problem. There are also a few studies that argue that optimal monetary policy may involve inflation, such as Wallace (2013). However, those papers study environments where either the distribution of money is non-degenerate or both money and credit are used. We abstract from these issues here and focus instead on how society responds to a given inflation rate and the social costs of such responses.

[12] The government is assumed to have enough coercive power to collect taxes in the CM, but has no coercive power in the DM. The government also cannot observe trading histories or asset holdings in neither the DM nor the CM. Hu, Kennan, and Wallace (2009) and Andolfatto (2010) consider alternative approaches to model deflation where the buyers can choose not to participate the CM in order to avoid paying taxes.

Figure 1: Timing of Representative Period



∞.[13] Figure 1 summarizes the timing of a representative period.

Given $\bar{e}$, the number of matches in the DM is determined by a constant-returns-to-scale matching function that depends on *market tightness*, defined as $\theta \equiv 1/\bar{e} \in [1, \infty]$, or the ratio of sellers to the *effective* buyers searching. A high θ implies a thick market for buyers and a thin one for sellers. Given θ, the meeting probability for an individual buyer with search intensity e is $e\alpha(\theta)$ while the meeting probability of a seller is $\alpha(\theta)/\theta$. The function $\alpha(\theta)$ satisfies $\alpha(\theta) \in [0, 1]$ for any $\theta \geq 1$ and is twice continuously differentiable, strictly increasing, strictly concave for $\theta \in [1, \infty)$, and satisfies the Inada conditions $\lim_{\theta\to\infty} \alpha(\theta) = 1$, $\lim_{\theta\to 1} \alpha(\theta) = 0$, $\lim_{\theta\to 1} \alpha'(\theta) \geq 1$, and $\lim_{\theta\to 1} \alpha(\theta)/\theta = 1$.

The instantaneous utility function of a buyer is

$$U^b(x, q, e) = u(q) - \psi(e) + x, \tag{1}$$

where q is consumption in the DM, x is the utility of consuming $x \in \mathbb{R}_+$ units of numéraire ($x < 0$ is interpreted as production), and e is the buyer's search effort.[14] We let $u(0) = 0$, $u'(0) = \infty$, $u'(q) > 0$, and $u''(q) < 0$ for $q > 0$. A buyer's lifetime expected utility is $\mathbb{E}_0 \left\{\sum_{t=0}^{\infty} \beta^t U^b(x_t, q_t, e_t)\right\}$, where $\mathbb{E}_0$ is the expectation operator conditional on time-0 information. The discount factor $\beta =$

[13] The assumption $\lim_{e\to 1} \psi'(e) = \infty$ ensures that the buyer's optimal search intensity is in $[0, 1]$.

[14] For tractability, the model requires that either the utility of consuming or the cost of producing the CM good is linear. In the formalization here, we simply assume that both CM consumption and production is linear though it would be straightforward to generalize to quasi-linear preferences $U(c) - y$, where c is CM consumption and y is CM production. Agents would then consume c^* in the CM where c^* satisfies $U'(c^*) = 1$. Normalizing $U(c^*) - c^*$ to zero would yield a model equivalent to the one presented here.

$(1+r)^{-1} \in (0,1)$ is the same for all agents and assumed to be smaller than γ throughout the analysis. Similarly, the instantaneous utility function of a seller is

$$U^s(x,q) = -c(q) + x, \tag{2}$$

where q is production in the DM and x is defined as before. We let $c(0) = c'(0) = 0$, $c'(q) > 0$, and $c''(q) \geq 0$. Further, we let $c(q) = u(q)$ for some $q > 0$ and denote by q^* the solution to $u'(q^*) = c'(q^*)$. Lifetime utility for a seller is given by $\mathbb{E}_0 \left\{ \sum_{t=0}^{\infty} \beta^t U^s(x_t, q_t) \right\}$.

3 Implementation

We study equilibrium outcomes that can be implemented by a mechanism designer in the DM called a *mechanism designer's proposal*. A *proposal* consists of four objects: *(i)* a sequence of functions in the bilateral matches, $o_t : \mathbb{R}_+^2 \to \mathbb{R}_+^3$, each of which maps the buyer's portfolio, (z_t, k_t), into a proposed allocation, $(q_t, d_{z,t}, d_{k,t}) \in \mathbb{R}_+ \times [0, z_t] \times [0, k_t]$, where q_t is the DM output produced by the seller and consumed by the buyer, $d_{z,t}$ is the transfer of real balances, and $d_{k,t}$ is the transfer of capital from the buyer to the seller; *(ii)* an initial distribution of money, μ; *(iii)* a sequence of prices for money, $\{\phi_t\}_{t=1}^{\infty}$, and a sequence of rental prices for capital, $\{R_t\}_{t=1}^{\infty}$, both in terms of the numéraire good; *(iv)* a sequence of measures of search intensity of buyers, $\{e_t\}_{t=1}^{\infty}$.

The trading procedure in the DM is given by the following game. Given agents' portfolio holdings and the associated proposed trade, both the buyer and the seller simultaneously respond with *yes* or *no*: if both respond with *yes*, then the planner proposed trade is carried out; otherwise, there is no trade. Since both agents can turn down the proposed trade, this ensures that trades are individually rational. We also require any proposed trade to be in the pairwise core.[15] Hence, the trading mechanism in pairwise meetings is chosen among all individually rational and renegotiation-proof mechanisms that maximize social welfare. Agents in the CM trade competitively against the proposed prices, which is consistent with the pairwise core requirement in the DM due to the equivalence between the core and competitive equilibria.

We denote s_b as the strategy of buyer $b \in \mathbb{B}$, which consists of three components for any given trading history h^t at the beginning of period t: *(i)* $s_b^{h^t,0}(z,k) = e \in \mathbb{R}_+$ that maps the buyer's portfolio, (z,k), into his search intensity, e, at the beginning of the DM; *(ii)* $s_b^{h^t,1}(z,k) \in \{yes, no\}$ that, contingent on being matched in the DM, maps the buyer's portfolio (z,k) to his *yes* or *no*

[15] The pairwise core requirement can be implemented directly with a trading mechanism that adds a renegotiation stage as in Hu, Kennan, and Wallace (2009). The renegotiation stage will work as follows. An agent will be chosen at random to make an alternative offer to the one made by the mechanism. The other agent will then have the opportunity to choose between the two offers.

response in the DM; *(iii)* $s_b^{h^t,2}(z,k,a_b,a_s) \in \mathbb{R}_+^2$ that maps the buyer's original portfolio, (z,k), and the buyer's and seller's choices whether to accept the trade, $a_b, a_s \in \{yes, no\}$, to his final real balances and capital holdings after the CM. The strategy of a seller $s \in \mathbb{S}$ at the beginning of period t consists of a function, $s_s^{h^t,1}(z,k) \in \{yes, no\}$, that represent the seller's response to trade contingent on the buyer's portfolio.

Definition 1. *An equilibrium is a list,* $\langle (s_b : b \in \mathbb{B}), (s_s : s \in \mathbb{S}), \mu, \{o_t, \phi_t, R_t, e_t\}_{t=1}^{\infty} \rangle$*, composed of one strategy for each agent and the proposal* $(\mu, \{o_t, \phi_t, R_t, e_t\}_{t=1}^{\infty})$ *such that: (*i*) each strategy is sequentially rational given other players' strategies; and (*ii*) the centralized market clears at every date.*

In what follows, we focus on stationary planner proposals where real balances are constant over time and equilibria such that *(i)* agents follow symmetric and stationary strategies; *(ii)* agents always respond with *yes* in all DM meetings; and *(iii)* the initial distribution of money is uniform across buyers. Following Hu, Kennan, and Wallace (2009), we call such equilibria *simple equilibria.* In a simple equilibrium, $\phi_t = \gamma\phi_{t+1}$ for all t; hence, we can discuss real balances only and leave out ϕ_t from a proposal. Moreover, the proposed DM trades, $o_t(z_t, k_t)$, are the same across time periods and can be written as $o(z,k) = [q(z,k), d_z(z,k), d_k(z,k)]$.

The outcome of a simple equilibrium is summarized by a list, $(q^p, d_z^p, d_k^p, z^p, k^p, e^p)$, where (q^p, d_z^p, d_k^p) is the terms of trade in the DM, e^p is the buyer's search intensity, and (z^p, k^p) are the portfolios of those buyers. Such an outcome, $(q^p, d_z^p, d_k^p, z^p, k^p, e^p)$, is said to be *implementable* if it is the equilibrium outcome of a simple equilibrium associated with a proposal $\{o, \mu, R, e\}$, and if for any portfolio (z,k), $o(z,k)$ selects a trade in the pairwise core. Given the proposals, we use $\mathcal{CO}(z,k;R)$ to denote the set of allocations in the pairwise core for each (z,k) given a rental price for capital, R.

For a given proposal, o, market thickness, θ, and rental price, R, let $V^b(z,k)$ and $W^b(z,k)$ denote the continuation values for a buyer holding (z,k) upon entering the DM and CM, respectively. Similarly, let $W^s(z,k)$ denote the continuation value for a seller holding (z,k) upon entering the CM. The problem for a buyer in the CM solves

$$W^b(z,k) = \max_{x,\hat{z}\geq 0,\hat{k}\geq 0} \left\{ x + \beta V^b(\hat{z},\hat{k}) \right\}$$

$$\text{s.t. } x + \gamma\hat{z} + \hat{k} = z + Rk + T$$

where $\hat{z}$ and $\hat{k}$ denotes the real balances and capital taken into the next DM and $T = (M_{t+1} - M_t)\phi_t$ is the lump-sum transfer of fiat money from the government. Since we focus on stationary equilibrium where real balances are constant over time, $\gamma = \frac{\phi_t}{\phi_{t+1}} = \frac{M_{t+1}}{M_t}$. Hence in order to

hold $\hat{z}$ real balances in the next period, the buyer must accumulate $\gamma\hat{z}$ units of real balances this period (since in a stationary equilibrium, the rate of return of fiat money is γ^{-1}). Substituting $x = z + Rk + T - \gamma\hat{z} - \hat{k}$ from the budget constraint, the Bellman equation for a buyer in the CM solves

$$W^b(z,k) = z + Rk + T + \max_{\hat{z}\geq 0, \hat{k}\geq 0} \left\{ -\gamma\hat{z} - \hat{k} + \beta V^b(\hat{z},\hat{k}) \right\}, \tag{3}$$

Due to linear preferences in the CM, note that (i) the buyer's value function is linear in total wealth, $W^b(z,k) = z + Rk + W^b(0,0)$, and (ii) the maximizing choice of $\hat{z}$ and $\hat{k}$ is independent of the buyer's current wealth, (z,k).

During the first stage, the value function of a buyer with portfolio (z,k) upon entering the DM, $V^b(z,k)$, is given by

$$V^b(z,k) = \max_{e\in[0,1]} \left\{ -\psi(e) + e\alpha(\theta) \left\{ u\left[q(z,k)\right] + W^b\left[z - d_z(z,k), k - d_k(z,k)\right] \right\} + [1 - e\alpha(\theta)]W^b(z,k) \right\}. \tag{4}$$

According to (4), a buyer searching with intensity e meets a seller with probability $e\alpha(\theta)$, consumes $q(z,k)$ and transfers to the seller $d_z(z,k)$ real balances and $d_k(z,k)$ units of capital. The buyer therefore enters the CM with $z - d_z(z,k)$ real balances and $k - d_k(z,k)$ units of capital. With probability $1 - e\alpha(\theta)$, a buyer is unmatched, in which case there is no trade in the DM. Using the linearity of $W^b(z,k)$, (4) simplifies to

$$V^b(z,k) = \max_{e\in[0,1]} \left\{ -\psi(e) + e\alpha(\theta) \left\{ u\left[q(z,k)\right] - d_z\left(z,k\right) - Rd_k\left(z,k\right) \right\} + W^b(z,k) \right\}. \tag{5}$$

For each portfolio (z,k), we use $e(z,k)$ to denote the optimal search intensity that solves the maximization problem in (5). Because ψ is strictly convex, it is straightforward to verify that $e(z,k)$ is uniquely defined and is continuous by the Theorem of Maximum. Moreover, when $(z,k) = (0,0)$, $e(z,k) = 0$. Noting that $\theta = 1/e^p$ in equilibrium, the buyer's choice of search intensity, $e^p = e(z^p,k^p)$, solves

$$-\psi'(e^p) + \alpha(1/e^p)\left[u(q^p) - d_z^p - Rd_k^p\right] = 0. \tag{6}$$

Substituting $V^b(z,k)$ with its expression given by (5) into (3), using the linearity of $W^b(z,k)$, and omitting constant terms, the buyer's portfolio problem in the CM can be reformulated as

$$\max_{(z,k)} \left\{ -iz - (1 + r - R)k - \psi(e(z,k)) + e(z,k)\alpha(\theta) \left\{ u\left[q(z,k)\right] - d_z\left(z,k\right) - Rd_k(z,k) \right\} \right\}, \tag{7}$$

where $i = \frac{\gamma-\beta}{\beta}$ is the cost of holding money and $1 + r - R$ is the cost of holding capital, which is the

difference between the gross rate of time preference and the rental price of capital.[16] In equilibrium, holding the equilibrium portfolio (z^p, k^p) is better than $(0, 0)$, and hence we must have

$$-iz^p - (1 + r - R)k^p - \psi(e^p) + e^p\alpha(1/e^p)[u(q^p) - d_z^p - Rd_k^p] \geq 0. \tag{8}$$

In a similar vein, the Bellman equation for a seller in the CM solves

$$W^s(z, k) = z + Rk + \max_{\hat{k}\geq 0}\left\{F(\hat{k}) - R\hat{k}\right\}. \tag{9}$$

According to (9), the seller's choice of input to operate the production technology is such that $F'(\hat{k}) = R$. Due to market clearing in the CM, the aggregate demand for capital must equal the aggregate supply: $k^p = \hat{k}$. Consequently, the equilibrium amount of capital stock, k^p, must satisfy

$$F'(k^p) = R \leq 1 + r. \tag{10}$$

According to (10), the equilibrium capital stock is such that the marginal product of capital, $F'(k^p)$, equals the rental rate of capital, R. It is also necessary that $R \leq 1 + r$. If $R > 1 + r$, buyers will hold an infinite amount of capital, but perfect competition implies that $F'(\infty) = 0 < 1 + r < R$, a contradiction. Moreover, using (9), the seller is willing to respond with *yes* to the proposed trade (q^p, d_z^p, d_k^p) only if

$$-c(q^p) + d_z^p + Rd_k^p \geq 0. \tag{11}$$

The above discussion shows that (6), (8), (10), and (11) are necessary conditions for an outcome $(q^p, d_z^p, d_k^p, z^p, k^p, e^p)$ to be implementable. In addition, we also impose the pairwise-core requirement. For any given rental price, R, and buyer's portfolio, (z^p, k^p), the pairwise core, $\mathcal{CO}(z^p, k^p; R)$, is defined as the set of all feasible allocations, $(q, d_z, d_k) \in \mathbb{R}_+ \times [0, z^p] \times [0, k^p]$, such that there are no alternative feasible allocations that would make both parties in the match better off, with at least one of the two being strictly better off. Requiring proposed trades to be in the pairwise core ensures that those trades are renegotiation-proof. A characterization of the pairwise core, $\mathcal{CO}(z^p, k^p; R)$, can be found in Hu and Rocheteau (2013)'s Supplementary Appendix B.

The following proposition shows that those conditions, together with the pairwise core requirement, are also sufficient.

[16] In this framework, it is equivalent to consider the effects of the money growth rate, inflation rate, or nominal interest rate due to the Fisher equation, $1 + i = \gamma(1 + r)$.

Figure 2: An Incentive-Feasible Mechanism

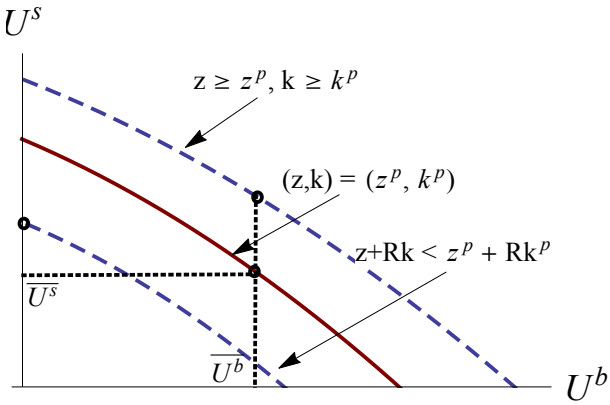
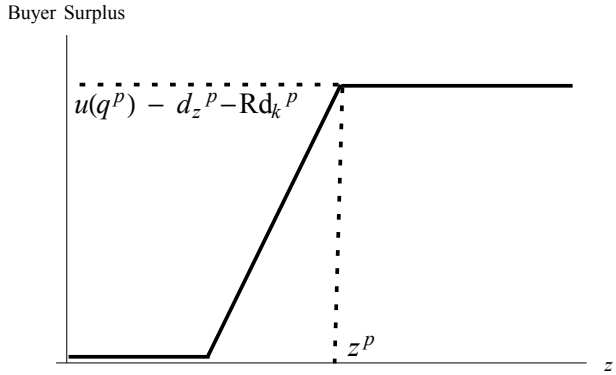


Figure 3: Buyer's Surplus Under Proposed Mechanism



Proposition 1. *An outcome,* $(q^p, d_z^p, d_k^p, z^p, k^p, e^p)$*, is implementable if and only if*

$$-iz^p - [1 + r - F'(k^p)]k^p + e^p\alpha(1/e^p)[u(q^p) - d_z^p - F'(k^p)d_k^p] - \psi(e^p) \geq 0, \tag{12}$$

$$d_z^p \leq z^p, \;\; d_k^p \leq k^p, \tag{13}$$

$$\psi'(e^p) = \alpha(1/e^p)[u(q^p) - d_z^p - F'(k^p)d_k^p], \tag{14}$$

$$-c(q^p) + d_z^p + F'(k^p)d_k^p \geq 0, \tag{15}$$

$$F'(k^p) \leq 1 + r, \tag{16}$$

and $(q^p, d_z^p, d_k^p) \in \mathcal{CO}(z^p, k^p; R)$.

Although the proof of Proposition 1 uses similar arguments to those in Hu and Rocheteau (2013), the set of implementable outcomes are quite different due to the additional choice of search intensity. The constraint (14) implies that in order to increase search intensity and hence the total matching probability, one has to give the buyer more surplus. While in Hu and Rocheteau (2013) it is always without loss of generality to give all the surplus to the buyers (at least for maximizing welfare), it is not the case here. In this setting, giving the buyer all the surplus restricts the equilibrium matching probability in a certain way, which may not be optimal at all or only optimal under certain inflation rates. In turn, the buyer's surplus can vary with the inflation rate, which provides a new channel for society to mitigate the cost of inflation more efficiently. This contrasts sharply with much of the literature that typically assumes (inefficient) trading protocols that are independent of the inflation rate, such as Nash bargaining or take-it-or-leave-it offers.

4 Optimal Allocation

In this section, we study implementable outcomes that are socially optimal. Formally, given an outcome, $(q^p, d_z^p, d_k^p, z^p, k^p, e^p)$, social welfare is defined as the discounted sum of buyers' and sellers' expected utilities:

$$\begin{aligned}\mathcal{W}(q^p, d_z^p, d_k^p, z^p, k^p, e^p) &= -k^p + \lim_{T\to\infty} \sum_{t=1}^{T} \beta^t \left\{ e^p \alpha \left(\frac{1}{e^p}\right) [u(q^p) - c(q^p)] - \psi(e^p) + [F(k^p) - k^p] \right\} \\ &= \frac{1}{r} \left\{ e^p \alpha \left(\frac{1}{e^p}\right) [u(q^p) - c(q^p)] - \psi(e^p) + [F(k^p) - (1+r)k^p] \right\}. \end{aligned} \tag{17}$$

The first term after the first equality is the utility cost incurred by agents in the initial CM to accumulate the proposed capital stock, k^p; the second term captures the utility flows in subsequent periods and consists of the sum of expected surpluses in pairwise meetings, $e^p\alpha(1/e^p)[u(q^p)-c(q^p)]$, the cost of searching, $\psi(e^p)$, and the output from the production technology net of the depreciated capital stock, $F(k^p) - k^p$.

Definition 2. *An outcome, $(q^p, d_z^p, d_k^p, z^p, k^p, e^p)$, is constrained efficient if it maximizes (17) subject to (12)–(16) and the pairwise core requirement.*

We begin with the benchmark case that maximizes social welfare (17) but without implementability constraints (12)–(16). The solution to this unconstrained problem, which we also call the *first-best allocation*, is given by $q^p = q^*$, $k^p = k^*$, and $e^p = e^*$ that solve

$$u'(q^*) = c'(q^*), \tag{18}$$

$$F'(k^*) = 1 + r, \tag{19}$$

$$\left[\alpha(1/e^*) - \alpha'(1/e^*)/e^*\right] [u(q^*) - c(q^*)] = \psi'(e^*). \tag{20}$$

A formal proof that these constitute the solution is given in the proof of Proposition 2. The first-best level of output, q^*, maximizes the match surplus between a buyer and seller, and the first-best level of capital, k^*, ensures that the marginal product of capital compensates for the opportunity cost of holding capital. The first-best level of search intensity, e^*, is derived from the first-order condition on the objective function with respect to e, but taking $q^p = q^*$. Accordingly, the marginal cost of searching, $\psi'(e^*)$, is equal to the corresponding social marginal contribution of searching, $[\alpha(1/e^*) - \alpha'(1/e^*)/e^*]$, times the surplus generated in each trade, $u(q^*) - c(q^*)$.

The remainder of this section is organized as follow. Section 4.1 first considers constrained-efficient outcomes for the economy without capital. In this case, Proposition 1 is still valid though

we have to set $k^p = 0$ and ignore (16). Section 4.2 studies constrained-efficient outcomes with both money and capital and compares the results with that of Section 4.1. Finally in Section 4.3, we consider an alternative formalization of the model that captures the endogenous search choice from the extensive margin.

4.1 Endogenous Search Intensity with Money Alone

Here we consider the economy without the production of capital, that is, we consider constrained-efficient outcomes with the additional constraint that $k^p = 0$ (and ignore (16)). The following lemma helps to characterize a constrained-efficient outcome without capital, (q^p, d_z^p, z^p, e^p).

Lemma 1. *Consider an economy with the constraint that $k^p = 0$. There exists z^p such that (q^p, d_z^p, z^p, e^p) is a constrained-efficient outcome if and only if the triple (q^p, d_z^p, e^p) solves*

$$\max_{(q,d_z,e)} e\alpha(1/e)[u(q) - c(q)] - \psi(e) \tag{21}$$

subject to

$$-id_z + e\alpha(1/e)[u(q) - d_z] - \psi(e) \geq 0, \tag{22}$$

$$\psi'(e) = \alpha(1/e)[u(q) - d_z], \tag{23}$$

$$-c(q) + d_z \geq 0. \tag{24}$$

Moreover, a solution to (21)– (24) exists, and any solution (q^p, d_z^p, e^p) satisfies $q^p \leq q^*$, $d_z^p \leq u(q^*)$, and $e^p \leq \hat{e}$ where $\hat{e}$ solves

$$\psi'(\hat{e})/\alpha(1/\hat{e}) = [u(q^*) - c(q^*)]. \tag{25}$$

Because of Lemma 1, we also call the triple (q^p, d_z^p, e^p) a constrained-efficient outcome if it solves (21)–(24). Constraints (23) and (24) are exactly the same as (14) and (15) but with $k^p = 0$. The constraint (22) differs from (12) in that it replaces z^p by d_z^p, and hence implicitly assumes $z^p = d_z^p$. As we show in the proof of Lemma 1, this assumption is satisfied when the first-best is not implementable, and, even when the first-best is implementable, there always exists a constrained-efficient outcome that satisfies this restriction. Moreover, note that there is no pairwise-core requirement in problem (21)–(24). It turns out that when maximizing social welfare, the pairwise-core requirement is not a binding constraint.

We are now ready to characterize the constrained-efficient outcomes for the economy with money alone.

Proposition 2. *Consider an economy with the constraint that $k^p = 0$. For any $i \geq 0$, a constrained-efficient outcome, $(q^p(i), d_z^p(i), e^p(i))$, exists, and satisfies the following.*

1. *Let*
$$i^* \equiv \frac{e^*\psi'(e^*) - \psi(e^*)}{u(q^*) - \psi'(e^*)/\alpha(1/e^*)}.$$
Then, $i^* > 0$ and for all $i \in [0, i^*]$, the unique constrained-efficient outcome, $(q^p(i), d_z^p(i), e^p(i))$, satisfies $q^p(i) = q^*$, $d_z^p(i) = d_z^* \equiv u(q^*) - \psi'(e^*)/\alpha(1/e^*)$, and $e^p(i) = e^*$.

2. *There exist $\bar{i} > i^*$ such that for all $i \in (i^*, \bar{i}]$, the unique constrained-efficient outcome, $(q^p(i), d_z^p(i), e^p(i))$, satisfies $q^p(i) < q^*$, $d_z^p(i) < d_z^*$, $e^p(i) > e^*$, and $\frac{d}{di}e^p(i) > 0$.*

3. *For any $e \in (0, 1]$, there exiss $i_e > i^*$ such that if $i > i_e$, then any constrained-efficient outcome, $(q^p(i), d_z^p(i), e^p(i))$, satisfies $q^p(i) < q^*$, $d_z^p(i) < d_z^*$, and $e^p(i) < e$.*

4. *Suppose that $\psi''(0) \in (0, \infty)$ and that for some $\delta > 0$, $\lim_{q\to 0}(c^{-1} \circ u)'(q)q^{0.5+\delta} > 0$. Then, for any i, equilibrium is monetary, that is, $d_z^p(i) > 0$. In addition, maximal welfare is strictly positive.*

Proposition 2 summarizes the effects of inflation in a pure currency economy without capital. The threshold $i^* > 0$ is the highest nominal interest rate such that the first-best is implementable. As a result, the Friedman rule, defined as $i = 0$, is sufficient but not necessary to achieve maximal welfare.[17] For all $i \in [0, i^*]$, money is superneutral and all welfare-relevant variables are at their first-best levels. While this superneutrality result also appears in earlier studies, a notable difference here is that the first-best cannot be implemented by giving all the surplus to the buyers. If this were the case, then under the first-best level of output, search intensity would be given by (23) with $q = q^*$, and hence equal to $\hat{e}$ given by (25). But due to search externalities, $\hat{e} > e^*$: by (20),

$$\psi'(e^*)/\alpha(1/e^*) < \psi'(e^*)/[\alpha(1/e^*) - \alpha'(1/e^*)/e^*] = [u(q^*) - c(q^*)] = \psi'(\hat{e})/\alpha(1/\hat{e}).$$

To discourage buyers from searching too much, the optimal mechanism only gives buyers a fraction of the surplus while the seller's individual rationality constraint, (24), is not binding at the optimum.

[17] This result was first shown by Hu, Kennan, and Wallace (2009) without endogenous search intensity and then subsequently extended by Hu and Rocheteau (2013) to incorporate capital. This finding however differs from the typical result in monetary models with exogenously given trading mechanisms such as pairwise bargaining where the Friedman rule is necessary for efficiency, at least with regards to the amount of output traded in a match. With endogenous participation or entry however, the Friedman rule need not be optimal. See also Rocheteau and Wright (2005) and Berentsen, Rocheteau, and Shi (2007) for a related discussion.

Proposition 2 shows that for moderate to high inflation rates, the first-best allocation is no longer incentive feasible. In fact, both DM output and search intensity deviate from their first-best levels when $i \in (i^*, \bar{i}]$. For nominal interest rates in this range, the buyer's search effort increases with inflation. While this result resembles the so-called "hot potato" effect of inflation, the underlying mechanism in our model differs from the conventional rationale. The standard explanation is that higher inflation itself induces buyers to search harder in order to get rid of their money holdings faster. However, this type of reasoning implicitly assumes a cash-in-advance constraint without which buyers may not hold cash in the first place. Instead, in our setting, the optimal mechanism dictates buyers to have higher surplus as inflation rises above i^*, which thereby induces buyers to search harder.

Here we give some intuition about why the optimal mechanism prescribes both the buyer's surplus and search effort to increase with inflation. When $i = i^*$, the optimal mechanism gives the buyers only a fraction of the total surplus to discourage them from over-searching and leaves the seller's participation constraint slack while the buyer's participation constraint binds. According to the first-order condition for search intensity, (14), the mechanism designer has two choices when inflation rises above i^*: either increase the buyer's surplus and increase search intensity, or decrease both. Given that both real output per match and real balances would decline with inflation, both choices may be good for overall welfare. However, given the buyer's participation constraint, (22), the second option would further tighten (22) while the first option would tend to relax it. As a result, the optimal mechanism always prescribes both the buyer's surplus and search effort to increase with inflation, at least locally.

Our finding that search efforts can rise with inflation contrasts sharply with previous studies with bilateral meetings and endogenous prices that study endogenous search decisions. Lagos and Rocheteau (2005) show that under Nash bargaining, the buyer's search effort always decreases with inflation. As this trading protocol is held constant for different inflation rates, both the buyer's real balances and the buyer's surplus fall with inflation, thereby inducing search efforts to also fall. More generally, buyers' search efforts are generically inefficient under pairwise bargaining, even at the Friedman rule, due to the usual congestion externality. This is not the case here since the first-best is implementable for a range of low inflation rates. In our model, exogenously imposing the Friedman rule or the Hosios condition is not necessary for implementing good allocations.

The above argument concerning the rise in search intensity is only valid when the seller's participation constraint, (24), is not binding at the optimum. Indeed the third part of Proposition 2 shows that search intensity can be arbitrarily small when the inflation rate is sufficiently high. This implies that the economy will eventually collapse into autarky as inflation rises. Although this result seems at odds with observations describing "hot potato" behavior in economies with

hyperinflation, we shall see in the next subsection that this eventual collapse would disappear when we introduce an alternative means of payments that does not suffer from the inflation tax.

We cannot give an explicit expression for the upper bound on the inflation rate below which the buyer's search intensity increases. While the proof for the first part of Proposition 2 only requires verification, the proof for the second part requires non-standard arguments. Because the objective function (21) is not concave and the feasible set is not convex, analytical solutions seem unavailable. Instead, we employ the Implicit Function Theorem to find a solution to the first-order conditions and use continuity to establish that the solution is also a global maximizer.

Nevertheless, we provide numerical examples to illustrate the possible effects of inflation in our model. We consider the functional forms $u(q) = \frac{(q+b)^{1-\sigma}-b^{1-\sigma}}{1-\sigma}$, $c(q) = \frac{q^{\kappa}}{\kappa}$, $\psi(e) = c\left(\frac{e}{1-e}\right)^{\rho}$, and $\alpha(\theta) = 1 - \exp(1-\theta)$ where $\theta = 1/e$. The parameter b ensures $u(0) = 0$, and with $b \approx 0$ implies relative risk aversion is constant at $\sigma q/(q+b) \approx \sigma \in (0,1)$. We choose the matching technology and cost function for search intensity to satisfy the technical assumptions given in Section 2. In our examples, we set $b = 0.0001$, $c = 0.4$, $\rho = 2$, $r = 0.02$ and report results for different values of σ and κ.[18]

In what follows, we define aggregate output as the total quantity of goods traded or production in the decentralized market:

$$\mathcal{Q} \equiv e\alpha(1/e)q,$$

where $e\alpha(1/e)$ is the buyer's matching probability, which is also equal to the frequency of trades in the DM.

Figures 5 – 10 assume $\sigma = 0.7$ and $\kappa = 1$ and plots output per match, search effort, aggregate output, the buyer's matching probability, real balances, and the buyer's surplus for a range of nominal interest rates. The threshold nominal interest rate, below which the first-best is implementable, is given by $i^* = 0.09$. Assuming each period corresponds to a year, this corresponds to a threshold inflation rate of $\gamma^* - 1 = (1+r)^{-1}(1+i^*) - 1 = 0.07$, or 7% annual inflation. In terms of welfare, low inflation is therefore costless as in Hu, Kennan, and Wallace (2009) and Rocheteau (2012).

Figure 10 plots the buyer's surplus and shows that for moderate inflation rates, the gains to trade that accrues to buyers increase with inflation. In turn, under the optimal mechanism, moderate inflation induces buyers to search harder and hence trade more frequently. This can be seen in Figures 6 and 8 which plot the buyer's search effort, e, and the frequency of DM trades, $e\alpha(1/e)$, respectively. While Figure 6 shows that search intensity remains above its first-best level

[18] While we do not present calibrated examples, we do investigate the sensitivity of our results to changes in parameters. Changing c or r does not affect much the main qualitative results, though obviously does affect e.g. the magnitude for the threshold nominal interest rate, i^*. For instance, increasing the scaling parameter for the cost function, c, tends to increase i^*. The examples are most sensitive to different values for σ, which controls the concavity DM utility function, and κ, which controls the convexity of the DM cost function.

even for very high inflation rates, we know from Proposition 2 that search effort will eventually fall towards zero as inflation tends to infinity. This eventual fall can be seen however in Figure 16 which plots the buyer's search intensity assuming $\sigma = 0.5$ and $\kappa = 1$.

We also find examples where aggregate output, or the total quantities traded in the DM, increases with inflation in some range. This can be seen in Figure 13 which plots aggregate output, $\mathcal{Q} = e\alpha(1/e)q$, assuming $\sigma = 0.7$ and $\kappa = 5$. For a range of moderate inflation rates, there are two opposing effects in our model: search intensity, e, and hence the frequency of trades, $e\alpha(1/e)$, increases with inflation while DM quantity traded per match, q, falls with inflation. In our examples, we find that the responsiveness of DM output to inflation is decreasing in the parameter κ, so that output is less responsive to inflation when $c(q)$ is more convex. Hence when κ is relatively large, it is possible for the total number of trades, $e\alpha(1/e)q$ to go up with inflation. This finding can thereby rationalize the non-linear effects of inflation on aggregate output documented by Bullard and Keating (1995) and Ahmed and Rogers (2000) for low-to-moderate inflation economies.[19]

On the surface, our findings on the non-monotonicities in search efforts and aggregate DM output seem similar to Lagos and Rocheteau (2005) under competitive price posting.[20] Lagos and Rocheteau (2005) find that search intensity can go up with inflation, but only for certain parameterizations and only for inflations rates close to the Friedman rule. In contrast, our finding that search intensity rises with inflation holds more generally for a range of inflation rates and is a feature of the optimal mechanism. That the the optimal mechanism delivers similar qualitative results as competitive search lends further credence to the non-monotonicities we identify. However, a direct comparison of the two mechanisms is not appropriate since the model with competitive search is a very different environment from the model with pairwise meetings we consider here. Nonetheless, a key difference is that the Friedman rule is necessary for efficiency under competitive search while that is clearly not the case here. Moreover, competitive search is not able to generate the coexistence of money and capital, a critical feature of our model that we turn to next.

[19] Bullard and Keating (1995) find that inflation may be positively correlated with long-run output using evidence from sixteen low inflation countries from 1960 to 1992. Similarly, Ahmed and Rogers (2000) find a positive correlation between long-run inflation and output using U.S. data from 1889 to 1995.

[20] Under price posting with partially directed search, sellers post prices while buyers direct their search toward a particular price. To attract buyers, sellers compete against one another and hence internalize the effect of inflation on the buyer's choice to hold money. Along the equilibrium path, competition partially compensates buyers for the detrimental effects of inflation by endogenously raising the buyer's share of the gains from trade. In this case, Lagos and Rocheteau (2005) show that buyers choose the socially efficient search intensity and real balances under the Friedman rule.

4.2 Endogenous Search Intensity with Money and Capital

Here we study constrained-efficient outcomes when both money and capital are present in the economy. Since the capital good may also be used as a medium of exchange, we will show that the economy may never collapse into autarky even for very high inflation rates. However, less obvious is the extent to which search intensity changes with inflation in the presence of capital. Nevertheless, we obtain a sufficient condition under which the buyer's search intensity increases with moderate inflation while the capital stock remains at its first-best level. However, there is no general guarantee that search intensity is bound to decrease as inflation tends to infinity.

Before considering the economy with both money and capital, it is useful to first consider the case where only capital is present as the liquid asset. This allows us to determine what is achievable with capital alone. Imposing the additional constraint that $z = 0$, an outcome may be denoted by (q, d_k, k, e). Such an outcome is implementable if and only if

$$-[1 + r - F'(k)]k + e\alpha(1/e)[u(q) - F'(k)d_k] \geq \psi(e), \tag{26}$$

$$-c(q) + F'(k)d_k \geq 0, \tag{27}$$

$$1 + r \geq F'(k), \tag{28}$$

$$\psi'(e) = \alpha(1/e)[u(q) - F'(k)d_k], \tag{29}$$

and $(q, d_k) \in \mathcal{CO}(0, k; R)$ with $R = F'(k)$.

Lemma 2. *Consider an economy with the constraint that $z = 0$. A constrained-efficient outcome, (q^c, d_k^c, k^c, e^c), exists. Moreover, the first-best is implementable if and only if*

$$(1 + r)k^* \geq u(q^*) - \frac{\psi'(e^*)}{\alpha(1/e^*)}.$$

When the first-best is not implementable, $d_k^c = k^c > k^*$. Moreover, the maximal social welfare given by

$$\mathcal{W}^c = \frac{1}{r}\left\{e^c\alpha(1/e^c)[u(q^c) - c(q^c)] - \psi(e^c) + F(k^c) - (1 + r)k^c\right\}$$

is strictly greater than what is achievable with the additional constraint that $k = k^*$, denoted $\mathcal{W}^0$.

Lemma 2 implies that the first-best allocation is implementable without money when the first-best capital stock, k^*, is sufficiently large. In that case, the aggregate capital stock is sufficiently abundant to allow buyers to finance consumption of the first-best. Since the first-best is implementable with $z^p = 0$, money is not essential.

When instead k^* is insufficient to meet the economy's liquidity needs, the optimal mechanism

features an overaccumulation of capital ($k^c > k^*$) in the absence of money. In addition, quantities traded in the DM are inefficiently low ($q^c < q^*$). With a shortage of liquidity, society faces a trade-off between two inefficiencies, as highlighted by Hu and Rocheteau (2013): (i) the shortage of capital for liquidity purposes, and (ii) the overaccumulation of capital for productive purposes. Accordingly, overaccumulation of capital may be socially optimal in order to mitigate the shortage of liquidity. Note that since it is always feasible to set $z = 0$, $\mathcal{W}^c$ gives a lower bound on welfare when both money and capital are present.

To illustrate these possibilities, we provide some numerical examples given in Table 1 for the economy with capital alone. These examples assume the functional forms given in the previous subsection plus $F(k) = Ak^a + (1-\delta)k$. We set $b = 0.0001$, $c = 0.4$, $\rho = 2$, $\kappa = 1$, $r = 0.02$, $a = 0.3$, $A = 0.8$, $\delta = 0.8$, and consider different values for $\sigma \in (0, 1)$.

Since money is not essential, output, search effort, and the capital stock are all independent of the cost of holding money, i. Table 1 summarizes values for constrained-efficient outcomes, (q^c, e^c, k^c), and the corresponding first-best allocations for two cases: $\sigma = 0.3$ and $\sigma = 0.7$. In both cases, the first-best is not implementable and hence there is overaccumulation of capital. However, the two cases differ as to how search intensity deviates from the first-best. When $\sigma = 0.3$, equilibrium search intensity is lower than the first-best level while for $\sigma = 0.7$, search intensity is higher than the first-best.

Table 1: Constrained-Efficient Outcomes with Capital Alone

	First-Best	$\sigma = 0.3$	First-Best	$\sigma = 0.7$
Output	$q^* = 1$	$q = 0.29$	$q^* = 1$	$q = 0.32$
Search Effort	$e^* = 0.22$	$e = 0.18$	$e^* = 0.34$	$e = 0.41$
Capital	$k^* = 0.17$	$k = 0.32$	$k^* = 0.17$	$k = 0.37$

We now turn to the case where both money and capital can serve as media of exchange. The next proposition characterizes the effects of inflation on allocations when there is a shortage of capital and shows how the buyer's search intensity varies with the cost of holding money. To simplify notation, we call a tuple $(q^p(i), z^p(i), k^p(i), e^p(i))$ a constrained-efficient outcome under a nominal interest rate i if there exists $(d_z^p, d_k^p) \leq (z^p(i), k^p(i))$ such that $(q^p(i), d_z^p, d_k^p, z^p(i), k^p(i), e^p(i))$ maximizes social welfare, (21), subject to the implementability constraints, (12)–(16).

Proposition 3. *Suppose that*

$$(1 + r)k^* < u(q^*) - \psi'(e^*)/\alpha(1/e^*).$$

For any $i \geq 0$, a constrained efficient outcome, $(q^p(i), z^p(i), k^p(i), e^p(i))$, exists, and satisfies the following.

1. *Let*
$$i^{**} = \frac{e^*\psi'(e^*) - \psi(e^*)}{u(q^*) - (1+r)k^* - \psi'(e^*)/\alpha(1/e^*)} > i^*.$$
For all $i \in [0, i^{**}]$, the constrained-efficient outcome, $(q^p(i), z^p(i), k^p(i), e^p(i))$, is unique, and satisfies $q^p(i) = q^*$, $z^p(i) > 0$, $k^p(i) = k^*$, and $e^p(i) = e^*$.

2. *Suppose that $1+r+F''(k^*)k^* < \frac{-F''(k^*)k^*}{i^{**}}$. There exist an $\bar{i}' > i^{**}$ such that for all $i \in (i^{**}, \bar{i}']$, the unique constrained-efficient outcome, $(q^p(i), z^p(i), k^p(i), e^p(i))$, satisfies $q^p(i) < q^*$, $k^p = k^*$, $z^p(i) > 0$, and $e^p(i) > e^*$. Moreover, $e^p(i)$ is strictly increasing in $i \in [i^{**}, \bar{i}']$.*

3. *There exist an $\hat{i}$ such that, for each $i > \hat{i}$, and for each constrained-efficient outcome, $(q^p(i), z^p(i), k^p(i), e^p(i))$, we have $k^p(i) > k^*$. Moreover, $z^p(i) \to 0$ as $i \to \infty$ but the maximum welfare converges to $\mathcal{W}^c > \mathcal{W}^0$.*

Proposition 3 assumes $(1+r)k^* < u(q^*) - \psi'(e^*)/\alpha(1/e^*)$ so that the first-best level of capital is insufficient to cover society's liquidity needs. Clearly, this assumption is necessary to make money essential.[21] Proposition 3 (1) shows that, similar to the pure monetary economy studied previously, the first-best is implementable for all $i \in [0, i^{**}]$. In this range, money is superneutral, in which case changes in inflation has no real effects on output or the capital stock.

For a range of intermediate inflation rates, Proposition 3 (2) gives a sufficient condition under which inflation has no effects on the capital stock even though output is inefficiently low and search intensity is inefficiently high. We remark here that monetary equilibrium exists, and hence money is essential, for a range of nominal interest rates above i^{**}, even without imposing the sufficient condition in part (2) (see Claim 2 in the proof of Proposition 3 part 2). As in Hu and Rocheteau (2013), we obtain rate-of-return dominance whenever both money and capital are used as media of exchange.

Proposition 3 (3) shows that capital overaccumulation is bound to occur as inflation rises. In turn, the monetary sector eventually collapses. Nevertheless, the economy never collapses into autarky as capital can always be used as a medium of exchange. Recall that without capital, agents eventually become less active in their search efforts as inflation gets very high. In the presence of capital however, search intensity can remain inefficiently high even at high inflation rates. Indeed, as welfare converges to the level where only capital is the medium of exchange, $\mathcal{W}^c$, Proposition 3

[21] The case where the economy faces a shortage of liquid assets is supported empirically by Caballero (2006) and Geanakopolos and Zame (2010) given a broader interpretation that capital serves both a means of payment and as collateral.

(3) suggests that search intensity also converges to its level without money, which may be higher or lower than e^*.

Here we provide some numerical examples to illustrate our findings. We assume the same functional forms as before with $b = 0.0001$, $c = 0.4$, $\rho = 2$, $\kappa = 1$, $r = 0.02$, $a = 0.3$, $A = 0.8$, $\delta = 0.8$, and consider different values for σ. Numerical examples illustrating the effects of inflation are summarized in Figures 19–24 for $\sigma = 0.7$ and Figures 31–36 for $\sigma = 0.3$.

Figure 20 plots the buyer's search intensity assuming $\sigma = 0.7$ and shows that search intensity remains above its first-best level and approaches the value reported in Table 1 for the economy with capital alone. In contrast, Figure 32 assumes $\sigma = 0.3$ and shows that search intensity eventually falls below its first-best level as inflation rises. As can be seen from Figure 20, the rise in search intensity can persist even for high inflation rates. These results are consistent with many recorded historical episodes of the "hot potato" effect of inflation.

For high inflation rates, the constrained-efficient allocation features both money and capital as means of payment, as can be seen in Figures 23 and 24 for $\sigma = 0.7$. In response, the optimal mechanism prescribes buyers to substitute money for capital as inflation increases. This Tobin effect turns out to be an optimal way of responding to inflation as doing so allows agents to maintain consumption in the DM even as inflation gets very high, as can be seen in Figure 19.

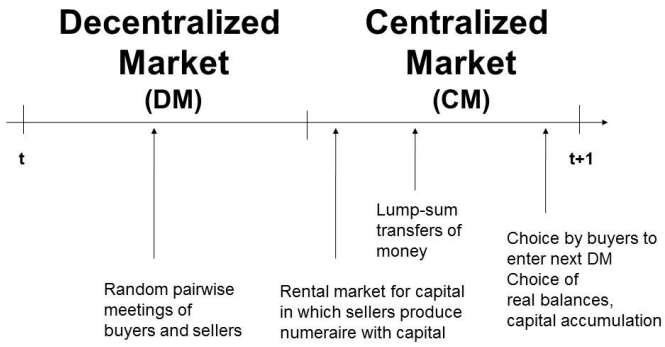
Moreover, our previous finding that aggregate output can rise with inflation also carries over to the model with both money and capital. Figure 27 plots aggregate output in the DM, assuming $\sigma = 0.7$ and $\kappa = 5$. As before, when output per trade is relatively unresponsive to inflation, it is possible for moderate inflation to induce an overall increase in the total number of DM trades.

4.3 Endogenous Participation

Here we consider an endogenous participation decision and study the effects of inflation along the extensive margin, similar to Liu, Wang, and Wright (2011).[22] We modify our baseline environment as follows. Instead of choosing search intensity, buyers can choose whether or not to enter the DM each date before the DM opens. If a buyer decides to enter, they incur a fixed cost $v > 0$ for doing so. We assume that this entry decision is made together with the portfolio decision in the previous period's CM since in any case, the buyer must take into account their entry decision when making their portfolio decision. Sellers enter for free, so that a unit measure of sellers always enter each period. The timing of a representative period is summarized in Figure 4.

[22] Using a mechanism design approach, Rocheteau (2012) considers a similar endogenous participation decision where agents can choose to be buyers or sellers in the DM. This formalization leads to similar results to the ones presented here. There are also many papers that study endogenous entry decisions in settings where the trading mechanism is taken such as Shi (1999), Rocheteau and Wright (2005), and Berentsen, Rocheteau, and Shi (2007).

Figure 4: Timing of Representative Period (Endogenous Participation)



Given the measure of buyers entering the DM, denoted n, market thickness is given by $\theta = 1/n$, and the buyer matching probability is given by $\alpha(\theta) = \alpha(1/n)$, where the function α satisfies the assumptions given in Section 2. Under the Inada conditions on α, we may assume that $n \in [0, 1]$. We also modify the production technology using the capital good and assume that $F(k) = Ak$ with $A < 1 + r$. This modification greatly simplifies the analysis and avoids issues such as whether or not buyers who do not enter the DM hold capital because the efficient amount of capital is zero.

As in Section 3, we study simple equilibria that can be implemented by a mechanism designer's proposal. Here, a proposal consists of (μ, o, ϕ, R, n), where the only new element is the proposed proportion of buyers entering the DM. The trading protocols are defined as before and the strategies and simple equilibria can be defined as before. An outcome then consists of $(q^p, d_z^p, d_k^p, z^p, k^p, n^p)$. We call such an outcome *implementable* if it is the equilibrium outcome of a simple equilibrium associated with a planner's proposal.

For a given proposal, o, market thickness, θ, and rental price, R, let $V^b(z, k)$ and $W^b(z, k)$ denote the continuation values for a buyer holding (z, k) upon entering the DM and CM, respectively, and let V_o^b denote the continuation value for a buyer who after the DM, stays outside the DM. Similarly, let $W^s(z, k)$ denote the continuation value for a seller holding (z, k) upon entering the CM. The value function of a buyer who decides to enter the DM and who holds (z, k) upon entering the DM, $V^b(z, k)$, solves

$$V^b(z, k) = \alpha(\theta) \left\{ u\left[q(z, k)\right] + W^b\left[z - d_z(z, k), k - d_k(z, k)\right] \right\} + [1 - \alpha(\theta)]W^b(z, k) - v, \qquad (30)$$

and the value function of a buyer with (z, k) upon entering the CM, $W^b(z, k)$, is given by

$$W^b(z,k) = z + Rk + \max\left\{\beta W^b(0,0), \max_{\hat{z}\geq 0, \hat{k}\geq 0}\left\{-\gamma\hat{z} - \hat{k} + T + \beta V^b(\hat{z}, \hat{k})\right\}\right\}, \tag{31}$$

Using the linearity of $W^b(z, k)$, (30) simplifies to

$$V^b(z,k) = \alpha(\theta)\left\{u\left[q(z,k)\right] - d_z\left(z,k\right) - Rd_k\left(z,k\right)\right\} + W^b(z,k). \tag{32}$$

Substituting $V^b(z, k)$ with its expression given by (32) into (31), using the linearity of $W^b(z, k)$, and omitting constant terms, the buyer's portfolio problem in the CM, together with their entry decision, can be reformulated as

$$\max\left\{0, \max_{(z,k)}\left\{-iz - (1+r-R)k - \alpha(\theta)\left\{u\left[q(z,k)\right] - d_z\left(z,k\right) - Rd_k(z,k) - v\right\}\right\}\right\}, \tag{33}$$

where $i = \frac{\gamma-\beta}{\beta}$ is the cost of holding money and $1 + r - R$ is the cost of holding capital.

In equilibrium, free entry of buyers implies that the above maximization problem in (33) should end up with a tie, that is, if $(q^p, d_z^p, d_k^p, z^p, k^p, n^p)$ is an equilibrium outcome, then

$$-iz^p - (1+r-R)k^p + \alpha(1/n^p)[u(q^p) - d_z^p - Rd_k^p] - v = 0. \tag{34}$$

Finally, following the same reasoning as in Section 3, we can conclude that in equilibrium, $R = A$. The following lemma characterizes implementable outcomes.

Lemma 3. *An outcome, $(q^p, d_z^p, d_k^p, z^p, k^p, n^p)$, with $n^p \in (0, 1)$ is implementable if and only if*

$$-iz^p - (1+r-A)k^p + \alpha\left(1/n^p\right)\left[u(q^p) - d_z^p - Ad_k^p\right] - v = 0, \tag{35}$$

$$d_z^p \leq z^p, \;\; d_k^p \leq k^p, \tag{36}$$

$$-c(q^p) + d_z^p + Ad_k^p \geq 0, \tag{37}$$

and $(q^p, d_z^p, d_k^p) \in \mathcal{CO}(z^p, k^p; A)$.

Given an outcome, $(q^p, d_z^p, d_k^p, z^p, k^p, n^p)$, social welfare is defined as the discounted sum of buyers' and sellers' expected utilities:

$$\mathcal{W}(q^p, d_z^p, d_k^p, z^p, k^p, n^p) \quad = \frac{1}{r}\left\{n^p\alpha\left(\frac{1}{n^p}\right)\left[u(q^p) - c(q^p)\right] - n^p v + n^p[Ak^p - (1+r)k^p]\right\}. \tag{38}$$

We say that an outcome is *constrained efficient* if it maximizes (38) subject to (35)–(37) and the

pairwise core requirement.

In this case, the first-best level of output, capital, and measure of buyers entering that maximize (38) (without considering participation constraints) is given by $(q^*, 0, n^*)$ and is such that

$$u'(q^*) = c'(q^*),$$

$$\left[\alpha\left(1/n^*\right) - \alpha'\left(1/n^*\right)/n^*\right]\left[u(q^*) - c(q^*)\right] = v. \tag{39}$$

Throughout this section we assume that

$$\alpha(1/n^*)[u(q^*) - c(q^*)] > v.$$

Note that without this assumption, the buyer is not willing to participate in the DM even at the first-best arrangement.

We first give a lemma that helps simplify our characterization of constrained-efficient outcomes.

Lemma 4. *Let $\bar{z}_i = \{[u(q^*) - c(q^*)] - v\}/i$ and let $\bar{k} = \{[u(q^*) - c(q^*)] - v\}/(1 + r - A)$. Then, There exists $(d_z^p, d_k^p) \leq (z^p, k^p)$ such that $(q^p, d_z^p, d_k^p, z^p, k^p, n^p)$ is a constrained-efficient outcome if the tuple (q^p, z^p, k^p, n^p) solves*

$$\max_{(q,z,k,n)\in[0,q^*]\times[0,\bar{z}_i]\times[0,\bar{k}]\times[0,n^*]} n\alpha(1/n)[u(q) - c(q)] - nv - n(1 + r - A)k \tag{40}$$

subject to

$$-iz - (1 + r - A)k + \alpha(1/n)[u(q) - z - Ak] - v = 0, \tag{41}$$

$$-c(q) + z + Ak \geq 0. \tag{42}$$

Moreover, if the first best allocation is not implementable, then for any constrained-efficient outcome, $(q^p, d_z^p, d_k^p, z^p, k^p, n^p)$, (q^p, z^p, k^p, n^p) solves (40)–(42).

Because of Lemma 4, we also refer to a solution of (40)–(42) as a constrained-efficient outcome. Before we present our characterization result, we first consider a benchmark case without fiat money, that is, we impose the constraint that $z^p = 0$. Under this additional constraint, the maximization problem becomes

$$\max_{(q,k,n)\in[0,q^*]\times[0,\bar{k}]\times[0,n^*]} n\alpha(1/n)[u(q) - c(q)] - nv - n(1 + r - A)k \tag{43}$$

subject to

$$-(1 + r - A)k + \alpha(1/n)[u(q) - Ak] - v = 0, \tag{44}$$

$$-c(q) + Ak \geq 0. \tag{45}$$

By the Extreme Value Theorem, a solution to problem (40)–(42) exists and the optimal value for welfare is unique, denoted by $\mathcal{W}^c$. We remark that $\mathcal{W}^c$ may be zero. It will also be useful to define a threshold for the fixed cost of entering the DM,

$$\bar{v} = \frac{\alpha(1/n^*)[c(q^*)(u(\bar{q}) - c(\bar{q})) - c(\bar{q})(u(q^*) - c(q^*))]}{c(q^*) - c(\bar{q})} > 0,$$

where $\bar{q} < q^*$ solves $u'(\bar{q})/c'(\bar{q}) = (1+r)/A$. We also define $\bar{A} = 1/(1+r)(1+i^*)$.

Proposition 4. *For any $i \geq 0$, a constrained efficient outcome, $(q^p(i), z^p(i), k^p(i), n^p(i))$, exists, and satisfies the following.*

1. *Let $i^* = \frac{\alpha(1/n^*)[u(q^*)-c(q^*)]-v}{c(q^*)} > 0$. Then, for all $i \in [0, i^*]$, the constrained-efficient outcome, $(q^p(i), z^p(i), k^p(i), n^p(i))$, is unique, and satisfies $q^p(i) = q^*$, $z^p(i) \geq c(q^*)$, $k^p(i) = 0$, and $n^p(i) = n^*$.*

2. *Suppose that $v < \bar{v}$ or that $A < \bar{A}$. There exists $\bar{i} > i^*$ such that for all $i \in (i^*, \bar{i}]$, the unique constrained-efficient outcome, $(q^p(i), z^p(i), k^p(i), n^p(i))$, satisfies $q^p(i) < q^*$, $z^p(i) = c(q^p)$, $k^p(i) = 0$, and $n^p(i) < n^*$. Moreover, $n^p(i)$ is strictly decreasing in $i \in (i^*, \bar{i}]$.*

3. *Suppose that $A = 0$. There exists $\bar{i}$ for which $i > \bar{i}$ implies that the constrained-efficient outcome is autarky.*

4. *Suppose that $\mathcal{W}^c > 0$. Then, if $i > \bar{i}$, any constrained-efficient outcome, $(q^p(i), z^p(i), k^p(i), n^p(i))$, satisfies $k^p(i) > 0$.*

Note that when $A = 0$, there is no capital production in the economy and hence is a special case of Proposition 4. For this case and for cases where A is sufficiently small (specifically, when $A < \bar{A}$), the findings in Proposition 4 resembles some aspects of Proposition 2. First, in both cases, the first-best is achievable for a range of low inflation rates. Second, in both cases, the buyer's matching probability ($e\alpha(1/e)$ for the intensive margin and $\alpha(1/n)$ for the extensive margin) increases with inflation. Under the extensive margin, this result is also similar to findings in Liu, Wang, and Wright (2011) and captures the idea that inflation makes it less likely for individuals to participate in monetary exchange. A key difference however is that here, a range of low nominal interest rates can implement the first-best, while this only occurs as a knife-edge case in Liu, Wang, and Wright (2011) where both the Friedman rule and Hosios condition are required for efficiency. Finally, when there is no capital, in both models the economy collapses into autarky for high inflation rates.

We also remark that $\mathcal{W}^c > 0$ when A is close to $1 + r$. When A is large and hence the use of capital is permitted, the findings in Proposition 4 also have similar predictions as Proposition 3.

First, under a sufficient condition ($v \leq \bar{v}$ in Proposition 4), the capital stock remains at its first-best level while the individual matching probability rises for a range of intermediate inflation rates. Second, capital overaccumulation is bound to occur for sufficiently high inflation rates. Together, our findings in Proposition 3 and Proposition 4 suggest the pattern that intermediate inflation affects the buyer's matching probability, but not capital accumulation. In addition, both models predict that the Tobin effect takes place only for high inflation rates.

There are also notable differences between the two models. First, recall that for low and intermediate inflation rates it is never optimal to give all the surplus to the buyers in the model with the intensive margin. However, it is always optimal to do so under the extensive margin with $A = 0$. Indeed, for any inflation rate, the constraint (35) is most relaxed when all the surplus goes to the buyer. Second, while there is a non-monotonic relationship between search intensity and inflation under the intensive margin, the second and third parts of Proposition 4 show that the number of buyers entering the DM always declines with inflation.

Here we present some numerical examples on the effects of inflation along the extensive margin. We consider the same functional forms as before, but with a linear technology for capital, $F(k) = Ak$. We set $b = 0.0001$, $\sigma = 0.5$, $\kappa = 1$, $r = 0.02$, $A = 0.9$, and $v = 0.4$. With $1 + r > A$, the first-best level of capital is given by $k^* = 0$.

Figures 37–42 illustrate the relationship between welfare-relevant variables and the nominal interest rate summarized in Proposition 4. From Figure 40, the buyer's matching probability increases monotonically with inflation when the cost of holding money is in an intermediate range. As with the intensive margin, the economy with capital is not guaranteed to collapse into autarky. This can be seen in Figures 37 and 38 where output and the measure of buyers both remain strictly positive even though they are inefficiently low. Accordingly, Figure 42 shows that welfare in the money and capital economy remains strictly positive even as inflation gets very high.

5 Concluding Remarks

This paper studies the consequences of anticipated inflation on economic exchange and welfare. We develop a tractable monetary model with three main features: costly search efforts to endogenize the frequency of trade, capital accumulation to endogenize the choice of a means of payment, and an endogenous trading mechanism that adjusts with the inflation tax. We revisit some classical issues in monetary economics, such as the long-run effects of inflation on output, search efforts, and capital accumulation as well as the social costs of inflation.

The model is able to replicate several qualitative patterns emphasized in both empirical macro studies and historical anecdotes, including monetary superneutrality for a range of low inflation

rates, non-linearities in trading frequencies and aggregate output, and substitution of money for capital for high inflation rates. While we acknowledge that certain aspects of our findings have appeared separately in previous studies, we show that they are intimately related by all being features of the optimal trading mechanism. Here we remark on a few caveats to our analysis and some important issues our paper abstracts from.

Optimal Trading Mechanism

In our framework, the economy's trading mechanism evolves to the optimal mechanism as the inflation rate changes. The inflation rate itself however is taken as exogenous. Since our focus is to study how trading arrangements adjust with changes in inflation, we take an admittedly partial equilibrium approach by not endogenizing the inflation rate. Importantly however, we do endogenize society's trading mechanism and obtain very different results from previous studies, most of which treat the trading mechanism as a primitive. Indeed we show that under the optimal mechanism, inflation has non-monotonic effects on matching probabilities and aggregate output in decentralized meetings. Moreover, we find that the hot potato effect, the coexistence of money and capital, and rate-of-return dominance are all optimal ways of responding to the inflation tax. That changes anticipated inflation can have severe consequences on economic exchange and social interactions has also been emphasized by various economic historians and anthropologists (Bresciani-Turroni (1931), Heynmann and Leijohnhufvud (1995), O'Dougherty (2002)).

Alternative Trading Mechanisms

In our baseline model, we assume that buyers and sellers meet pairwise in the first stage and look for individually rational and coalition-proof mechanisms that maximize social welfare. Alternatively, we could consider other mechanisms in the first stage that still satisfy the core requirement. One example is Walrasian price-taking as in Lucas and Prescott (1974) where individuals meet in large groups and trade against market-clearing prices. In the Appendix, we show that a version of our model with competitive pricing in the first stage also delivers non-monotone search intensity.[23] Due to the equivalence between competitive equilibrium and the core, this arrangement is still consistent with the core requirement.

However, there are some notable differences compared with our baseline model. First, under competitive pricing, search intensity can rise with inflation but only near the Friedman rule and only if the seller's production cost is strictly convex. Since a strictly convex production cost delivers marginal cost pricing, the buyer's surplus can increase with inflation so long as the cost of holding

[23] We thank Guillaume Rocheteau for making this suggestion to us.

money is sufficiently small. Second, the competitive equilibrium is generically inefficient since with ex-post competition, the congestion externality is not internalized: while the Friedman rule delivers the first-best level of output, search intensity is either too high or too low. Consequently, it is possible for inflation can increase welfare near the Friedman rule if search intensity is inefficiently low. Finally, while competitive pricing can generate a non-monotonic matching probability for certain parameters, we conjecture that it cannot deliver rate-of-return dominance for any inflation rate. While money and capital can coexist under Walrasian pricing, there will be rate-of-return equality, similar to Lagos and Rocheteau (2008).

Other Substitutes for Domestic Currency

In the model, we assume that capital goods are the only alternative means of payments to money. However, capital goods in the model can be interpreted more broadly to include other real assets that may provide a hedge against inflation. This includes the use of assets not only for immediate settlement but as collateral (Caballero (2006)).[24] An example is the use of home equity as collateral to finance future consumption (Mian and Sufi (2011)). Moreover, individuals often resort to using foreign currencies for transactions during periods of high or hyperinflation (Calvo and Vegh (1992)). While our current framework cannot fully accommodate for the circulation of foreign currencies, an extension of our model to multiple countries and currencies is a fruitful topic for future research. Such a model could then determine how the presence of foreign currencies affects the consequences of inflation on international trade and welfare (Zhang (2014)).

Other Uses of Seigniorage Revenue

Here the seigniorage revenue generated by an increase in the money supply is used to give lump-sum transfers to private citizens. Some papers in the literature consider alternative uses of seigniorage revenue, such as subsidizing the unsecured credit sector. The implications to the real economy would obviously depend on how the seigniorage revenue is used, and, by assuming lump-sum transfers and quasi-linear preferences, we are examining the worst case for welfare cost of inflation. Indeed, inflation could have positive effects on welfare when distribution of money holdings is not degenerate (Wallace (2013), Rocheteau, Weill, and Wong (2014)), or when the credit sector is present and the seigniorage revenue is used to finance private debt (Araujo and Hu (2014)).

[24] The role of assets as collateral also appears in Kiyotaki and Moore (2008) where assets do not change hands along the equilibrium path. This would entail DM trades using secured credit with capital playing the role of collateral. Then in the CM, debtors would settle obligations in numeraire. In our current set-up, capital goods are transferred between individuals and there is finality in each DM trade.

6 Figures

Endogenous Search Intensity With Money Only ($\sigma = 0.7$, $\kappa = 1$)

Figure 5: Output per Match

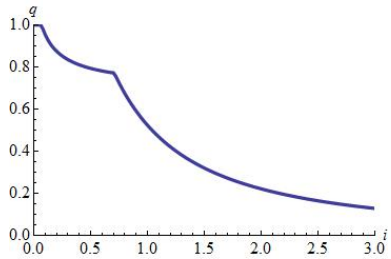


Figure 6: Search Intensity

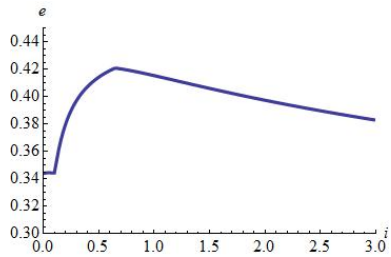


Figure 7: Aggregate Output

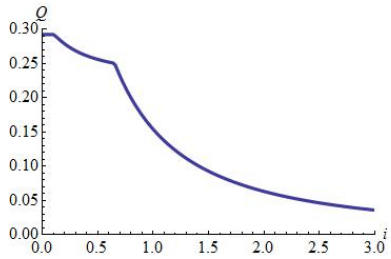


Figure 8: Matching Probability

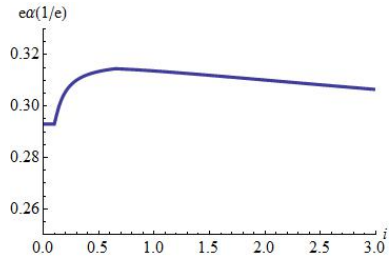


Figure 9: Real Balances

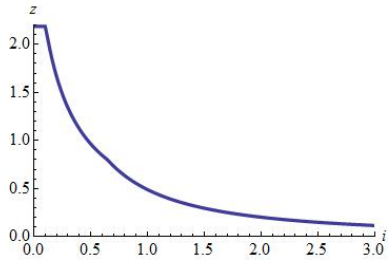
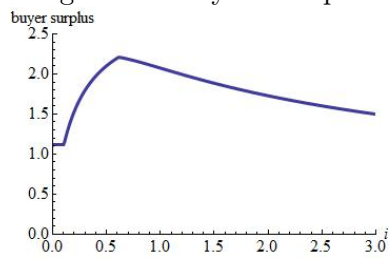


Figure 10: Buyer's Surplus



Endogenous Search Intensity With Money Only ($\sigma = 0.7$, $\kappa = 5$)

Figure 11: Output per Match

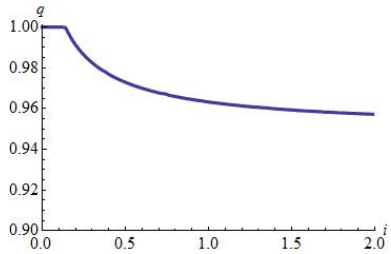


Figure 12: Search Intensity

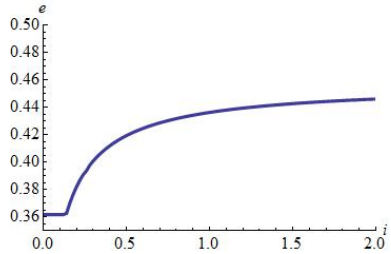


Figure 13: Aggregate Output

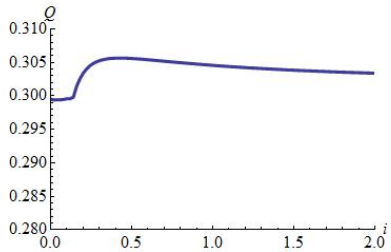
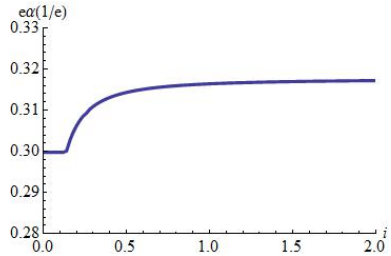


Figure 14: Matching Probability



Endogenous Search Intensity With Money Only ($\sigma = 0.5$, $\kappa = 1$)

Figure 15: Output per Match

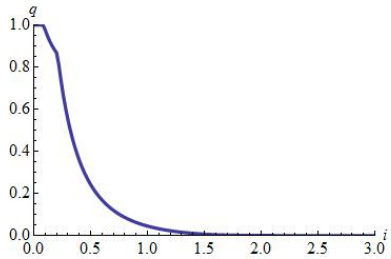


Figure 16: Search Intensity

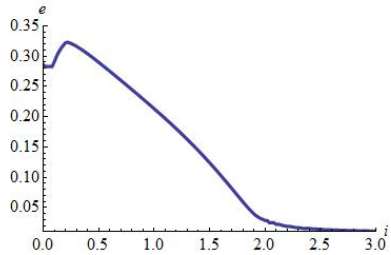


Figure 17: Aggregate Output

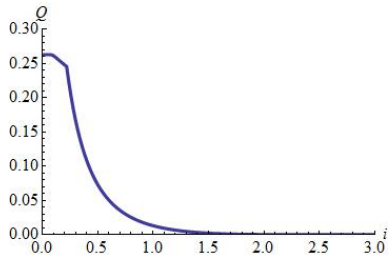
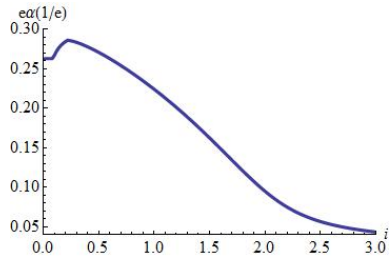


Figure 18: Matching Probability



Endogenous Search Intensity With Money and Capital ($\sigma = 0.7$, $\kappa = 1$)

Figure 19: Output per Match

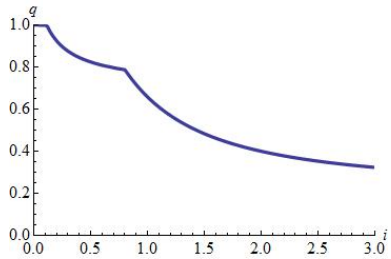


Figure 20: Search Intensity

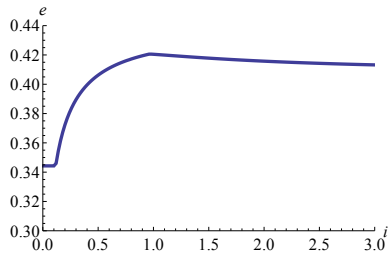


Figure 21: Aggregate Output

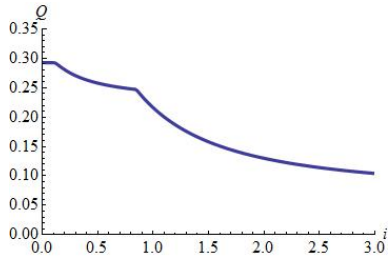


Figure 22: Matching Probability

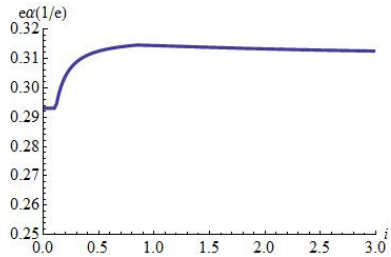


Figure 23: Real Balances

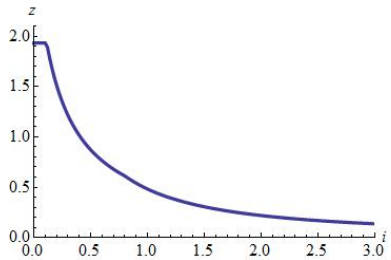
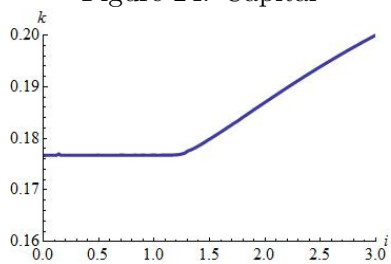


Figure 24: Capital



Endogenous Search Intensity With Money and Capital ($\sigma = 0.7$, $\kappa = 5$)

Figure 25: Output per Match

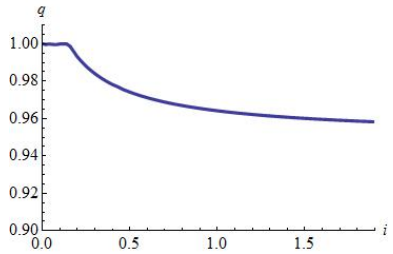


Figure 26: Search Intensity

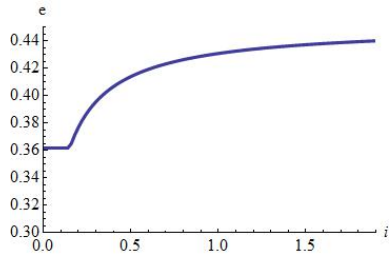


Figure 27: Aggregate Output

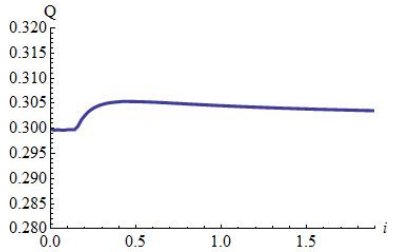


Figure 28: Matching Probability

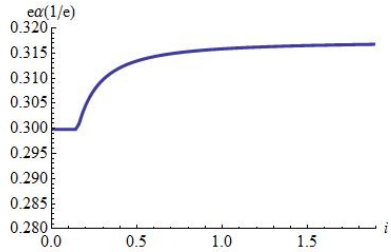


Figure 29: Real Balances

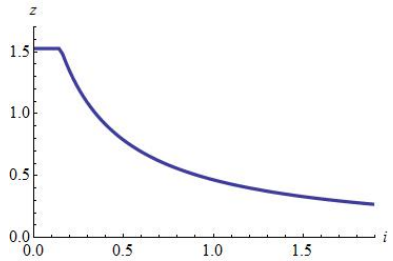
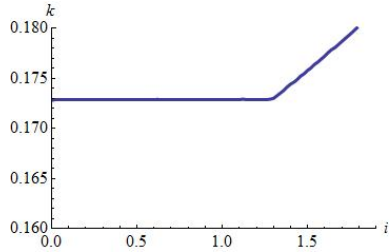


Figure 30: Capital



Endogenous Search Intensity With Money and Capital ($\sigma = 0.3$, $\kappa = 1$)

Figure 31: Output per Match

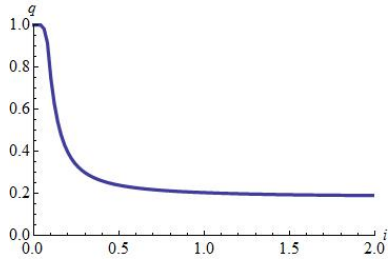


Figure 32: Search Intensity

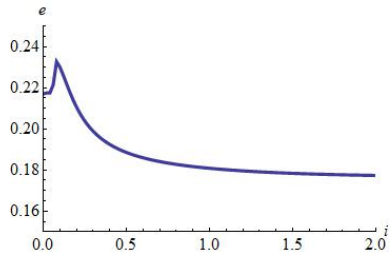


Figure 33: Aggregate Output

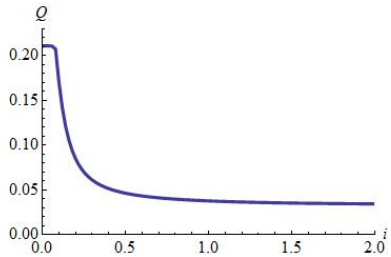


Figure 34: Matching Probability

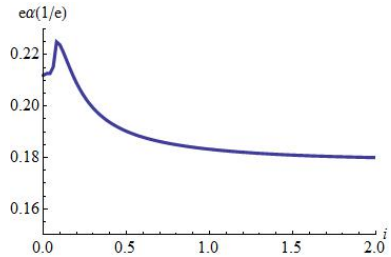


Figure 35: Real Balances

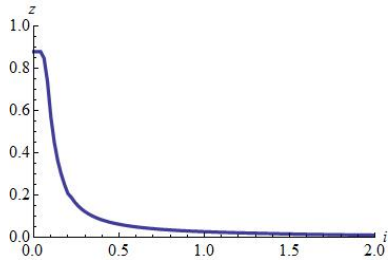
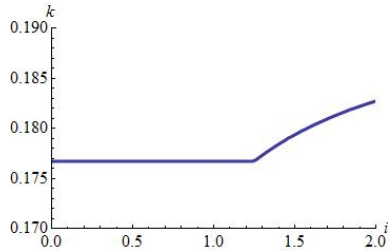


Figure 36: Capital



Endogenous Participation with Money and Capital ($\sigma = 0.5$, $\kappa = 1$)

Figure 37: Output per Match

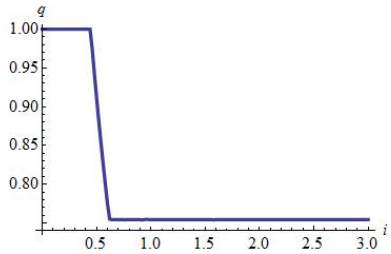


Figure 38: Measure of Buyers

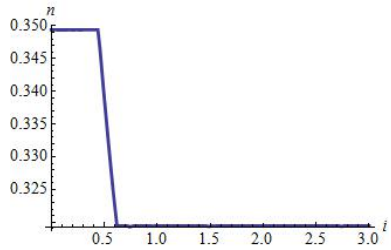


Figure 39: Aggregate Output

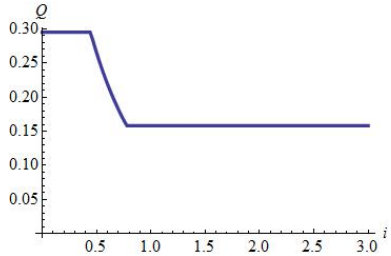


Figure 40: Matching Probability

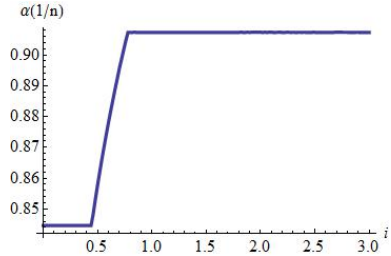


Figure 41: Capital

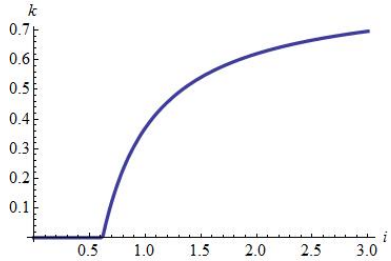
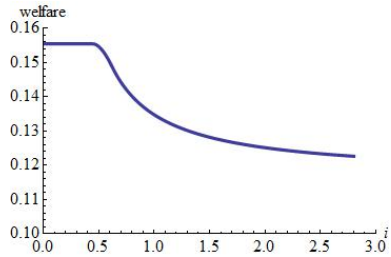


Figure 42: Welfare



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Appendix

Proof of Proposition 1

We proved the necessity of constraints (12)-(16) in the main text. Here we prove their sufficiency. Let $(q^p, d_z^p, d_k^p, z^p, k^p, e^p)$ be an outcome that satisfies (12)-(16) and the pairwise core requirement. Consider the following trading mechanism with $R = F'(k^p)$:

1. If $(z, k) \geq (z^p, k^p)$, then

$$\begin{aligned} o(z,k) &\in \arg\max_{q,d_z,d_k} \{d_z + Rd_k - c(q)\} \\ \text{s.t.} \quad & u(q) - d_z - Rd_k \geq u(q^p) - d_z^p - Rd_k^p, \\ & q \geq 0, \ d_z \in [0, z], \ d_k \in [0, k]. \end{aligned} \tag{46}$$

2. Otherwise,

$$\begin{aligned} o(z,k) &\in \arg\max_{q,d_z,d_k} \{d_z + Rd_k - c(q)\} \\ \text{s.t.} \quad & u(q) - d_z - Rd_k = 0, \\ & q \geq 0, \ d_z \in [0, z], \ d_k \in [0, k]. \end{aligned} \tag{47}$$

Solutions to the maximization problems (46) and (47) exist, and are denoted by $o(z, k) = [q(z, k), d_z(z, k), d_z(z, k)]$. Each solution has a unique $q(z, k)$. Although d_z and d_k may not be uniquely determined, we select the solution such that $d_z(z, k) = z$ if it exists and $d_k(z, k) = 0$ otherwise for any $(z, k) \neq (z^p, k^p)$. To show that (q^p, z^p, k^p) is a solution to (46) for $(z, k) = (z^p, k^p)$, notice that (46) is the dual problem that defines the core of a pairwise meeting. Because $(q^p, z^p, k^p) \in \mathcal{CO}(z^p, k^p; R)$, it is also a solution to (46). This gives us a well-defined mechanism, o.

Now we show that the following strategy profile, (s_b^*, s_s^*), form a simple equilibrium: for all t and for all h^t, $(s_b^*)^{h^t,0}(z, k) = e^p$ if $(z, k) \geq (z^p, k^p)$, $(s_b^*)^{h^t,0}(z, k) = 0$ otherwise; for all portfolios (z, k), $(s_b^*)^{h^t,1}(z, k) = yes$ for all portfolios (z^p, k^p) and all responses (a_b, a_s), and $(s_s^*)^{h^t,1}(z, k) = yes$; $(s_b^*)^{h^t,2}(z, k, a_b, a_s) = (z^p, k^p)$. In words, irrespective of their portfolios when entering the CM, buyers exit the CM with their proposed portfolios, (z^p, k^p). The effort choice is e^p if the buyer holds no less than the proposed portfolio in both assets; it is zero otherwise. In the DM they always say *yes* to the proposals. We show that s_b^* and s_s^* are optimal strategies following any history, given that all other agents follow (s_b^*, s_s^*).

Conditions (12) and (15), as well as the constraints in (46) and (47), ensure that both buyers and sellers are willing to respond with *yes* to the mechanism, both on and off equilibrium paths.

Now, by (46) and (47), the buyer's surplus is given by

$$u\left[q(z,k)\right] - d_z\left(z,k\right) - Rd_k(z,k) = u(q^p) - d_z^p - Rd_k^p \text{ if } (z,k) \geq (z^p,k^p); \tag{48}$$
$$u\left[q(z,k)\right] - d_z\left(z,k\right) - Rd_k(z,k) = 0 \text{ otherwise.}$$

As a result, because e^p satisfies (14) and $R = F'(k^p)$, it follows that $e(z,k) = e^p$ if $(z,k) \geq (z^p,k^p)$ and $e(z,k) = 0$ otherwise. Now consider the problem (7). By (48), any choice (z,k) with $(z,k) \neq (z^p,k^p)$ are strictly dominated by $(0,0)$. Finally, (z^p,k^p) is better than $(0,0)$ by (12). □

Proof of Lemma 1

Under the constraint that $k^p = 0$, by Proposition 1, a constrained-efficient outcome, (q^p, d_z^p, z^p, e^p), solves

$$\max_{(q,d_z,z,e)} e\alpha(1/e)[u(q) - c(q)] - \psi(e) \tag{49}$$

subject to

$$-iz + e\alpha(1/e)[u(q) - d_z] - \psi(e) \geq 0, \tag{50}$$
$$\psi'(e) = \alpha(1/e)[u(q) - d_z], \tag{51}$$
$$-c(q) + d_z \geq 0, \tag{52}$$
$$d_z \leq z, \tag{53}$$

and the pairwise core requirement.

Call the problem (49)-(53) program $\mathcal{A}$ and (21)-(24) program $\mathcal{B}$.

Assume (q^p, d_z^p, z^p, e^p) is a solution to $\mathcal{A}$. As $z^p \geq d_z^p$ the following is true:

$$-iz^p + e^p\alpha(1/e^p)[u(q^p) - d_z^p] - \psi(e^p) \leq -id_z^p + e^p\alpha(1/e^p)[u(q^p) - d_z^p] - \psi(e^p), \tag{54}$$

that is (q^p, d_z^p, e^p) satisfies the constraints in program $\mathcal{B}$. Now, if (q', d_z', e') has a higher value than that of (q^p, d_z^p, e^p), then, because (q', d_z', z', e') with $z' = d_z'$ also satisfies the constraints (pairwise core because $d_z' = z'$) in program $\mathcal{A}$, it also has a higher value than (q^p, d_z^p, z^p, e^p), a contradiction. Thus, (q^p, d_z^p, e^p) solves $\mathcal{B}$ as well.

Conversely, suppose that (q^p, d_z^p, e^p) solves program $\mathcal{B}$. Then, by setting $z^p = d_z^p$, (22) implies (12), and hence (q^p, d_z^p, z^p, e^p) satisfies the constraints in $\mathcal{A}$. If there is another $(q', d_z', z', e,)$ that

gives a higher value than (q^p, d^p, z^p, e^p), then (q', d', e'), which also satisfies the constraints in $\mathcal{B}$, also gives a higher value than (q^p, d_z^p, e^p), a contradiction. Thus, (q^p, d_z^p, z^p, e^p) solves $\mathcal{A}$ as well.

We now show that any solution, (q^p, d_z^p, e^p), to (21)-(24) is such that $q^p \leq q^*$, $d_z^p \leq u(q^*)$, and $e^p \leq \hat{e}$ where $\hat{e}$ is given by (25).

To show that $q^p \leq q^*$, suppose by contradiction that $q^p > q^*$. Then the planner could decrease q^p to q^* to increase trade surplus while changing the transfer of real balances to $d_z' = u(q^*) - u(q^p) + d_z^p \geq c(q^*)$ such that this deviation is incentive compatible. Since we have assumed $q^* < q^p$, we have $c(q^*) < c(q^p)$ and $u(q^*) - c(q^*) > u(q^p) - c(q^p)$. Then,

$$u(q^*) - c(q^*) \geq u(q^p) - d_z^p \leq u(q^p) - c(q^p).$$

Hence $q^p \leq q^*$.

With $q^p \leq q^*$, it follows from the buyer's participation constraint, (50), that $d_z^p \leq u(q^*)$. Then from (51),

$$\psi'(e^p)/\alpha(1/e^p) = u(q^p) - d_z^p \leq u(q^*) - c(q^*).$$

Hence $e^p \leq \hat{e}$ such that $\hat{e}$ solves (25).

As a result, we may impose the constraints $q^p \leq q^*, d_z^p \leq u(q^*)$, and $e^p \leq \hat{e}$ with no loss in generality. Moreover with these additional constraints, (21)–(24) is a maximization problem with a continuous objective function and a compact feasible set. Hence a solution exists. □

Proof of Proposition 2

(1) To determine the first-best allocation, we examine the unconstrained problem (21) without constraints (22), (23), and (24). The first-best allocation, (q^*, e^*), solves

$$u'(q^*) = c'(q^*),$$

$$\left[\alpha(1/e^*) - \alpha'(1/e^*)/e^*\right][u(q^*) - c(q^*)] = \psi'(e^*).$$

Given (q^*, e^*), we can obtain d_z^* from (22):

$$d_z^* \equiv u(q^*) - \psi'(e^*)/\alpha(1/e^*).$$

Here we show that the first-best is unique. For any $q \in \mathbb{R}$, let $e(q)$ be the unique e that solves

$$\frac{\partial}{\partial e}\left\{e\alpha(1/e)[u(q) - c(q)] - \psi(e)\right\} = 0.$$

Moreover, it is straightforward to verify, using the Envelope Theorem, that if

$$u(q') - c(q') < u(q) - c(q),$$

then $e(q') < e(q)$ and

$$e(q')\alpha[1/e(q')][u(q') - c(q')] - \psi[e(q)] < e(q')\alpha[1/e(q)][u(q) - c(q)] - \psi[e(q)].$$

Hence, for any $(q, e) \neq (q^*, e^*)$, either $q \neq q^*$, or $q = q^*$ but $e \neq e^*$, and so

$$e\alpha(1/e)[u(q) - c(q)] - \psi(e) < e^*\alpha(1/e^*)[u(q^*) - c(q^*)] - \psi(e^*).$$

Now we show that for all $i \in [0, i^*]$, the first-best solution (q^*, e^*, d_z^*) satisfies constraints (22), (23), and (24). Note that (23) holds by construction, and hence

$$\alpha(1/e^*)[u(q^*) - d_z^*] = \psi'(e^*).$$

Plugging this into (22), it is straightforward to verify that (22) holds if and only if

$$i \leq i^* = \frac{e^*\psi'(e^*) - \psi(e^*)}{u(q^*) - \psi'(e^*)/\alpha(1/e^*)}.$$

Finally, (24) holds if and only if

$$u(q^*) - \psi'(e^*)/\alpha(1/e^*) \geq c(q^*),$$

that is,

$$\alpha(1/e^*)[u(q^*) - c(q^*)] \geq \psi'(e^*).$$

But, by (20) and the fact that $\alpha'(\theta) > 0$ for all θ,

$$\alpha(1/e^*)[u(q^*) - c(q^*)] = \alpha'(1/e^*)/e^*[u(q^*) - c(q^*)] + \psi'(e^*) > \psi'(e^*).$$

(2) Here we will consider the maximization problem (21) subject to (22) and (23) at equality, but without (24). Claim 1 below justifies the assumption regarding the binding participation constraint for the buyer. It also shows the non-binding constraint for the seller at i^* and later we show that it is also non-binding locally.

Claim 1. (i) When $i = i^*$, the seller's participation constraint, (24), holds with strict inequality

at the optimum, (q^*, d_z^*, e^*); (ii) when $i > i^*$, the buyer's participation constraint, (22), binds at the optimum; (iii) when $i > i^*$, $q^p < q^*$ at the optimum.

Proof. **(i)** We have shown it in (1).

(ii) To show that (22) binds for all $i > i^*$, consider the Lagrangian associated with (21), (22), (23), (24), $q \geq 0$, $d_z \geq 0$, and $e \geq 0$:

$$\begin{aligned}
\mathcal{L}(q, d_z, e; \lambda, \mu, \eta, \nu_q, \nu_z, \nu_e) &= e\alpha(1/e)[u(q) - c(q)] - \psi(e) \\
&+ \lambda\{-id_z + e\alpha(1/e)[u(q) - d_z] - \psi(e)\} \\
&+ \mu\{d_z - c(q)\} \\
&+ \eta\{\psi'(e) - \alpha(1/e)[u(q) - d_z]\} \\
&+ \nu_q q + \nu_z d_z + \nu_e e,
\end{aligned}$$

where $\lambda \geq 0$, $\mu \geq 0$, η, $\nu_q \geq 0$, $\nu_z \geq 0$, and $\nu_e \geq 0$ are the Lagrange multipliers associated with (22), (24), (23), $q \geq 0$, $d_z \geq 0$, and $e \geq 0$ respectively. From the Kuhn-Tucker Theorem, the following are the first-order necessary conditions taken with respect to q, d_z, e and the complementary slackness conditions for $q \geq 0$, $d_z \geq 0$, and $e \geq 0$:

$$e^p\alpha(1/e^p)[u'(q^p) - c'(q^p)] + \lambda e^p\alpha(1/e^p)u'(q^p) - \mu c'(q^p) - \eta\alpha(1/e^p)u'(q^p) + \nu_q = 0, \tag{55}$$

$$[\alpha(1/e^p) - \alpha'(1/e^p)/e^p][u(q^p) - c(q^p)] - \psi'(e^p) + \lambda\{[\alpha(1/e^p) - \alpha'(1/e^p)/e^p][u(q^p) - d_z^p] - \psi'(e^p)\} + \nu_e$$
$$= -\eta\{\psi''(e^p) + [\alpha'(1/e^p)/(e^p)^2][u(q^p) - d_z^p]\}, \tag{56}$$

$$\lambda[i + e^p\alpha(1/e^p)] = \mu + \eta\alpha(1/e^p) + \nu_z, \tag{57}$$

$$\nu_q q^p = 0, \tag{58}$$

$$\nu_z d_z^p = 0, \tag{59}$$

$$\nu_e e^p = 0. \tag{60}$$

In addition, (22) and (24) are not binding only if $\lambda = 0$ and $\mu = 0$, respectively.

To verify that (22) binds for all $i > i^*$, i.e. $\lambda > 0$, suppose by contradiction that $\lambda = 0$. Then (22) holds with strict inequality and hence, $q^p > 0$ and $e^p > 0$. From (24) and $q^p > 0$, we have $d_z^p > 0$. Then from (58), (59), and (60), we have $\nu_q = 0$, $\nu_z = 0$, and $\nu_e = 0$. Combining (55) and (57) with $\nu_q = 0$ and $\nu_z = 0$ yield

$$\frac{u'(q^p)}{c'(q^p)} = \frac{e^p\alpha(1/e^p) + \mu}{e^p\alpha(1/e^p) + \mu - \lambda i}. \tag{61}$$

From (61) and $\lambda = 0$, $q^p = q^*$. And from (57), $\lambda = 0$, and $\nu_z = 0$, we have $-\eta\alpha(1/e^p) = \mu$. If $\mu = 0$, then $\eta = 0$, and from (56) and $\nu_e = 0$, we have $e = e^*$, a contradiction. Now suppose that $\mu > 0$. Then, (24) is binding and hence $d_z^p = c(q^*)$. By (57) and $\nu_z = 0$, we have $\mu = -\eta\alpha(1/e^p) > 0$. But then by (56), this implies

$$[\alpha(1/e^p) - \alpha'(1/e^p)/e^p][u(q^*) - c(q^*)] - \psi'(e^p) > 0,$$

and hence $e^p < e^*$. However by (23),

$$\psi'(e^p) = \alpha(1/e^p)[u(q^*) - c(q^*)],$$

and, because $d_z^p = c(q^*) < d_z^*$, this implies $e^p > e^*$. This leads to a contradiction. Hence $\lambda > 0$ and therefore (22) is binding.

(iii) Here we prove that $q^p < q^*$ for $i > i^*$. From (61), $q^p \neq q^*$ unless $\lambda = 0$, which is violated when $i > i^*$. Now suppose that $q^p > q^*$. Consider the following deviation: the planner could decrease q^p to q^* to increase trade surplus while changing the transfer of real balances to $d_z' = d_z^p - [u(q^p) - u(q^*)] \geq c(q^*)$ such that this deviation is incentive compatible. This deviation would raise welfare and is incentive feasible, a contradiction. Hence $q^p < q^*$. □

Observe that with (22) at equality and (23), the maximization problem simplifies to a choice of e for a given i. Given i and a choice of e, (22) at equality and (23) implies a unique solution for q and d_z as functions of e and i:

$$d_z(e,i) = \frac{e\psi'(e) - \psi(e)}{i}; \tag{62}$$

$$q(e,i) = u^{-1}\left(\frac{e\psi'(e) - \psi(e)}{i} + \frac{\psi'(e)}{\alpha(1/e)}\right). \tag{63}$$

Let

$$g(e,i) \equiv \frac{e\psi'(e) - \psi(e)}{i} + \frac{\psi'(e)}{\alpha(1/e)}$$

and $f(x) = u^{-1}(x)$. Then $f'(g(e,i)) = \frac{1}{u'[f(g(e,i))]} > 0$ for all (e,i). Moreover,

$$1 - c'[f(g(e,i))]f'(g(e,i)) > 0 \text{ if } q < q^*$$

and

$$1 - c'[f(g(e,i))]f'(g(e,i)) = 0 \text{ if } q = q^*.$$

The objective function can be written as

$$G(e,i) = e\alpha(1/e)\left\{g(e,i) - c\left[f(g(e,i))\right]\right\} - \psi(e).$$

Claim 2. There exists an $\bar{i} > i^*$ such that for all $i \in [i^*, \bar{i}]$, there is a unique maximizer $e^p(i)$ for $\max_{e\in[0,1]} G(e,i)$ and satisfies $\frac{d}{de}e^p(i) > 0$ for all $i \in [i^*, \bar{i}]$. Moreover, $(q(e^p(i),i), e^p(i), d_z(e^p(i),i))$ is the unique constrained-efficient outcome for all $i \in [i^*, \bar{i}]$.

Proof. The proof applies the Implicit Function Theorem (IFT).

First we show that

$$\frac{\partial^2}{\partial e^2}G(e^*,i^*) < 0. \tag{64}$$

The first partial derivative of $G(e,i)$ with respect to e is

$$\begin{aligned}
\frac{\partial G(e^*,i^*)}{\partial e} &= \alpha(1/e^*)\left[g(e^*,i^*) - c\left[f(g(e^*,i^*))\right]\right] - \frac{e^*\alpha'(1/e^*)}{e^{*2}}\left[g(e^*,i^*) - c\left[f(g(e^*,i^*))\right]\right] \\
&+ e\alpha(1/e^*)\left[\frac{\partial g(e^*,i^*)}{\partial e} - c'[f(g(e^*,i^*))]f'(g(e^*,i^*))\frac{\partial g(e^*,i^*)}{\partial e}\right] - \psi'(e^*) \\
&= \left[\alpha(1/e^*) - \frac{\alpha'(1/e^*)}{e^*}\right]\left[g(e^*,i^*) - c\left[f(g(e^*,i^*))\right]\right] \\
&+ e^*\alpha(1/e^*)\frac{\partial g(e^*,i^*)}{\partial e}\underbrace{\left[1 - c'[f(g(e^*,i^*))]f'(g(e^*,i^*))\right]}_{=0 \text{ at } q=q^*} - \psi'(e^*) \\
&= 0.
\end{aligned}$$

The second partial derivative is

$$\begin{aligned}
\frac{\partial^2 G(e^*,i^*)}{\partial e^2} &= \left[\alpha(1/e^*) - \frac{\alpha'(1/e^*)}{e^*}\right]\frac{\partial g(e^*,i^*)}{\partial e}\underbrace{\left[1 - c'[f(g(e^*,i^*))]f'(g(e^*,i^*))\right]}_{=0 \text{ if } q=q^*} \\
&+ \underbrace{\left[g(e^*,i^*) - c\left[f(g(e^*,i^*))\right]\right]}_{(+) \text{ since } u(q^*)-c(q^*)>0}\underbrace{\frac{\alpha''(1/e^*)}{e^{*3}}}_{(-) \text{ since } \alpha''<0} - \underbrace{\psi''(e^*)}_{(+) \text{ since } \psi''>0} \\
&+ \underbrace{e\alpha(1/e^*)\left[\frac{\partial g(e^*,i^*)}{\partial e}\right]^2}_{(+)}\underbrace{\left[-c''[f'(g(e^*,i^*))]^2 - c'[f(g(e^*,i^*))f''\right]}_{(-)} \\
&< 0.
\end{aligned}$$

This verifies that $\frac{\partial^2}{\partial e^2}G(e^*,i^*) < 0$.

Then, because $\frac{\partial}{\partial e}G(e^*, i^*) = 0$ by Claim 1 (i), (64), and by the IFT, there is an open neighborhood $(e_0, e_1) \times (i_0, i_1)$ around (e^*, i^*) and a continuously differentiable implicit function $e^p : (i_0, i_1) \to (e_0, e_1)$ such that for all $i \in (i_0, i_1)$, $e^p(i)$ is the unique $e \in (e_0, e_1)$ such that

$$\frac{\partial}{\partial e}G(e^p(i), i) = 0,$$

and that $\frac{\partial^2}{\partial e^2}G(e, i) < 0$ in that neighborhood (note that G is continuously twice differentiable). This shows that $G(e, i)$ is locally concave, and, as the unique solution to the first-order conditions, $e^p(i)$ is the local maximizer in that neighborhood.

To show that it is also the global maximizer, first consider $M(i) = \max_{e \notin (e_0, e_1)} G(e, i)$. By the Theorem of the Maximum in e.g. Stokey and Lucas (1989), $M(i)$ is continuous and $M(i^*) < G(e^*, i^*)$. Let $\delta = G(e^*, i^*) - M(i^*) > 0$. Then by continuity, there exists an $i_2 \in (i^*, i_1]$ such that if $i \in [i^*, i_2]$, then $M(i) \leq M(i^*) + \delta/3 < G(e^*, i^*) - \delta/3 \leq G(e^p(i), i)$. Hence, for all $i \in [i^*, i_2]$, $e^p(i)$ maximizes $G(\cdot, i)$.

Because the function $-c(q(e, i)) + d_z(e, i)$ is continuous and because $-c(q^*) + d_z^* \geq 0$, it follows from continuity that there exists an $i_3 \in (i^*, i_2]$ such that for all $i \in (i^*, i_3]$, $-c(q(e^p(i), i)) + d_z(e^p(i), i) \geq 0$. Now we show that $(q(e^p(i), i), e^p(i), d_z(e^p(i), i))$ is the unique constrained-efficient outcome for $i \in (i^*, i_3]$.

Suppose that (q', e', d'_z) solves (21)-(24). By Claim 1, (q', e', d'_z) satisfies (22) at equality and hence $q' = q(e', i)$ and $d'_z = d(e', i)$. Because $G(e, i)$ is the same objective function as (21) after substituting q by $q(e, i)$ and d_z by $d_z(e, i)$, it follows that $G(e', i) \leq G(e^p(i), i)$. But, because (q', e', d'_z) is constrained efficient, $G(e', i) = G(e^p(i), i)$. Moreover, because we have a unique maximizer for $\max_e G(e, i)$, it follows that $e' = e^p(i)$ and hence $q' = q(e^p(i), i)$ and $d'_z = d_z(e^p(i), i)$. Thus, for each $i \in (i^*, i_3]$, $(q^p, e^p, d_z^p) = (q(e^p(i), i), e^p(i), d_z(e^p(i), i))$ is the constrained-efficient outcome.

Finally, we show that the derivative of $e^p(i)$ is positive locally. By IFT again, $e^p(i)$ is continuously differentiable, and for all $i \in (i^*, i_3]$, $\frac{d}{di}e^p(i)$ is given by

$$\frac{d}{di}e^p(i) = -\frac{\partial^2}{\partial e \partial i}G(e^p(i), i) / \frac{\partial^2}{\partial e^2}G(e^p(i), i). \tag{65}$$

Now we claim that at i^*,

$$\frac{\partial^2}{\partial e \partial i}G(e^*, i^*) > 0. \tag{66}$$

To compute (66), we differentiate to obtain

$$\begin{aligned}
\frac{\partial^2}{\partial e \partial i} G(e^*, i^*) &= \alpha(1/e^*) \frac{\partial}{\partial i} g(e^*, i^*) \underbrace{\{1 - c'[f(g(e^*, i^*))] f'(g(e^*, i^*))\}}_{= 0 \text{ if } q = q^*} \\
&+ e^* \alpha'(1/e^*)(-1/e^{*2}) \frac{\partial}{\partial i} g(e^*, i^*) \underbrace{\{1 - c'[f(g(e^*, i^*))] f'(g(e^*, i^*))\}}_{= 0 \text{ if } q = q^*} \\
&+ e^* \alpha(1/e^*) \frac{\partial^2}{\partial e \partial i} g(e^*, i^*) \underbrace{\{1 - c'[f(g(e^*, i^*))] f'(g(e^*, i^*))\}}_{= 0 \text{ if } q = q^*} \\
&+ e^* \alpha(1/e^*) \frac{\partial}{\partial i} g(e^*, i^*) \{-c''[f(g(e^*, i^*))][f'(g(e^*, i^*))]^2 - c'[f(g(e^*, i^*))] f''\} \frac{\partial}{\partial e} g(e^*, i^*) \\
&= e^* \alpha(1/e^*) \underbrace{\frac{\partial}{\partial i} g(e^*, i^*)}_{(-)} \{- \underbrace{c''[f(g(e^*, i^*))][f'(g(e^*, i^*))]^2}_{(+)} - \underbrace{c'[f(g(e^*, i^*))] f''}_{(+)}\} \underbrace{\frac{\partial}{\partial e} g(e^*, i^*)}_{(+)} \\
&> 0.
\end{aligned}$$

This verifies that $\frac{\partial^2}{\partial e \partial i} G(e^*, i^*) > 0$. Hence because $e^p(i)$ is continuously differentiable, there exist an $\bar{i} \in (i^*, i_3]$ such that for all $i \in [i^*, \bar{i}]$, $\frac{d}{di} e^p(i) > 0$. By (65) (recall that we have established that $\frac{\partial^2}{\partial e^2} G(e, i) < 0$ locally), $e^p(i) > e^*$ for all such i's.

To show that $d_z^p < d_z^*$ for $i \in (i^*, \bar{i}]$, we have from (23),

$$d_z^p = u(q^p) - \psi'(e^p)/\alpha(1/e^p) < u(q^*) - \psi'(e^*)/\alpha(1/e^*) = d_z^*,$$

since $q^p < q^*$ and $e^p(i) > e^*$. □

By Claim 2, for each $i \in (i^*, \bar{i}]$, $(q(e^p(i), i), e^p(i), d_z(e^p(i), i))$ is the unique constrained-efficient outcome. Moreover, we have $\frac{d}{di} e^p(i) > 0$.

(3) By (23), $d_z^p = u(q^p) - \psi'(e^p)/\alpha(1/e^p)$. Thus, we may rewrite (22) and (24) as

$$\frac{\psi'(e^p)}{\alpha(1/e^p)} + \frac{e^p \psi'(e^p) - \psi(e^p)}{i} \geq u(q^p), \tag{67}$$

$$u(q^p) - c(q^p) \geq \frac{\psi'(e^p)}{\alpha(1/e^p)}. \tag{68}$$

By Lemma 1, for any constrained-efficient outcome, $e^p(i) \leq \hat{e} < 1$.

Fix some $e \in (0, \hat{e}]$. Let q_e satisfy

$$u(q_e) - c(q_e) = \psi'(e)/\alpha(1/e).$$

Since $\psi'(e)/\alpha(1/e)$ is increasing in e and continuous for $e \in (0,1)$, it follows that $q_e \in (0, q^*]$ is uniquely determined and varies continuously in e. Let

$$i(e) = \frac{e\psi'(e) - \psi(e)}{u(q_e) - \psi'(e)/\alpha(1/e)}.$$

Then, $i(e) \in (0, \infty)$ and is continuous in e. Now we show that if $i > i(e)$, then (e, q) does not satisfy (67) and (68) with respect to i for any q. Suppose, by contradiction, that (e, q) satisfies (67) and (68) with respect to i. Then, by (68),

$$u(q) - c(q) \geq \psi'(e)/\alpha(1/e) = u(q_e) - c(q_e),$$

and hence, $q \geq q_e$. But by (67),

$$u(q) \leq \frac{\psi'(e)}{\alpha(1/e)} + \frac{e\psi'(e) - \psi(e)}{i} < \frac{\psi'(e)}{\alpha(1/e)} + \frac{e\psi'(e) - \psi(e)}{i(e)} = u(q_e),$$

which implies that $q < q_e$, a contradiction.

Finally, for each $e \in (0, \hat{e}]$, let

$$i_e = \max\{i(e') : e' \in [e, \hat{e}]\}.$$

Notice that i_e is well-defined because $i(e)$ is continuous and $[e, \hat{e}]$ is a compact set. Now, if $i > i_e$, then for any $e' \in [e, \hat{e}]$, $i > i_{e'}$ and hence (e', q) does not satisfy (67) and (68) with respect to i for any q. Thus, $e^p(i) < e$.

(4) Recall that $d_z(e, i) = \frac{e\psi'(e) - \psi(e)}{i}$. Define $h(e) = \frac{\psi'(e)}{\alpha(1/e)}$. Then, $h' > 0$, $h(0) = 0$, $h(1) = \infty$, $\frac{\partial}{\partial e} d_z > 0$, $d_z(0, i) = 0$, and $d_z(1, i) = \infty$.

Recall also that $q(e, i) = u^{-1}(h(e) + d_z(e, i))$. Then, (q, e) satisfies (67) if $q = q(e, i)$ and it satisfies (68) if, in addition to $q = q(e, i)$,

$$u(q) - c(q) \geq h(e),$$

that is, if

$$h(e) + d_z(e, i) - c \circ u^{-1}(h(e) + d_z(e, i)) \geq h(e),$$

which is equivalent to

$$c^{-1} \circ u(d_z(e, i)) \geq h(e) + d_z(e, i).$$

First we show that for e close to 1,

$$c^{-1} \circ u(d_z(e,i)) < h(e) + d_z(e,i). \tag{69}$$

To see this, notice that because $c^{-1} \circ u$ is concave and because of the Inada conditions,

$$\lim_{e \to 1} \frac{c^{-1} \circ u(d_z(e,i))}{d_z(e,i)} = 0$$

and hence, for e sufficiently large,

$$\frac{c^{-1} \circ u(d_z(e,i))}{d_z(e,i)} < 1 < 1 + \frac{h(e)}{d_z(e,i)}.$$

Now we show that for e sufficiently small,

$$c^{-1} \circ u(d_z(e,i)) > h(e) + d_z(e,i). \tag{70}$$

First notice that $\frac{\partial}{\partial e} d_z(e,i) = e\psi''(e)/i$ and

$$h'(e) = \frac{\psi''(e)\alpha(1/e) + \psi'(e)\alpha'(1/e)/e^2}{\alpha(1/e)^2}.$$

Let $\psi''(0) = A$, which, by assumption, is strictly positive. Then,

$$\lim_{e \to 0} h'(e) = \frac{A + (\psi'(e)/e)(\alpha'(1/e)/e)}{\alpha(1/e)^2} \in [A, 2A],$$

where $\lim_{e \to 0} \alpha(1/e) = 1$, $\lim_{e \to 0} \alpha'(1/e)/e \leq \lim_{e \to 0} \alpha(1/e) = 1$, and $\lim_{e \to 0} \psi'(e)/e = \psi''(0) = A$. Thus, for sufficiently small e, $d_z(e,i) \in (eA/2i, 2Ae/i)$ and $h'(e) \in (A/2, 4A)$. Thus, for such e's,

$$d_z(e,i) \in ((A/4i)e^2, (A/i)e^2) \text{ and } h(e) \in ((A/2)e, (4A)e),$$

and hence, for such e's,

$$d_z(e,i) + h(e) < (4A)e + (A/i)e^2.$$

By assumption, there exists a δ for which

$$\lim_{q \to 0} (c^{-1} \circ u)'(q) q^{0.5+\delta} > 0.$$

Therefore, for sufficiently small q, $(c^{-1} \circ u)'(q) > Kq^{-0.5-\delta}$ for some $K > 0$, and hence $c^{-1} \circ u(q) >$

$Kq^{0.5-\delta}$ for all such q's. Thus, for e sufficiently small,

$$c^{-1} \circ u(d_z(e,i)) \geq K d_z(e,i)^{0.5-\delta} > ((A/4i)e^2)^{0.5-\delta} \equiv Le^{1-2\delta}.$$

Because $\lim_{e\to 0} \frac{Le^{1-2\delta}}{(4A)e+(A/i)e^2} = \infty$, it follows that, for e sufficiently small,

$$c^{-1} \circ u(d_z(e,i)) > Le^{1-2\delta} > (4A)e + (A/i)e^2 \geq h(e) + d_z(e,i).$$

This proves (70).

Now, by (69) and (70), and by the Intermediate Value Theorem, there exists $\tilde{e}_i > 0$ such that

$$d_z(\tilde{e}_i, i) = c \circ u^{-1}(h(\tilde{e}_i) + d_z(\tilde{e}_i, i)).$$

Then, $(q(\tilde{e}_i, i), \tilde{e}_i)$ satisfies (67) and (68). Moreover, the outcome $(q(\tilde{e}_i, i), \tilde{e}_i)$ is associated with positive welfare given by $\mathcal{W}(i)$. By (23),

$$\alpha(1/\tilde{e}_i)[u(q(\tilde{e}_i, i)) - c(q(\tilde{e}_i, i))] = \psi'(\tilde{e}_i)$$

and hence welfare is

$$\mathcal{W}(i) = \tilde{e}_i \alpha(1/\tilde{e}_i)[u(q(\tilde{e}_i, i)) - c(q(\tilde{e}_i, i))] - \psi(\tilde{e}_i) = \tilde{e}_i \psi'(\tilde{e}_i) - \psi(\tilde{e}_i) > 0.$$

Proof of Lemma 2

Here we show that the first-best allocation without fiat money ($z^p = 0$) is implementable if and only if $(1+r)k^* \geq u(q^*) - \frac{\psi'(e^*)}{\alpha(1/e^*)}$.

(1) Here we show only sufficiency and necessity is proved in (2) below. Suppose that $(1+r)k^* \geq u(q^*) - \frac{\psi'(e^*)}{\alpha(1/e^*)}$ and consider the first-best outcome $(q^c, d_k^c, k^c, e^c) = (q^*, d_k^*, k^*, e^*)$, where

$$\alpha(1/e^*)[u(q^*) - (1+r)d_k^*] = \psi'(e^*).$$

Since $F'(k^*) = 1+r$, (26) and (28) are satisfied. Note that $d_k^* \leq k^*$ because $(1+r)k^* \geq u(q^*) - \frac{\psi'(e^*)}{\alpha(1/e^*)}$ and (29) is satisfied by construction. The proof that it satisfies (27) follows the same argument as in the proof of Proposition 1 (1).

(2) Suppose that $(1+r)k^* < u(q^*) - \frac{\psi'(e^*)}{\alpha(1/e^*)}$. Here we show that $k^0 > k^*$ and hence the first-best is not implementable and that $\mathcal{W}^c > \mathcal{W}^0$.

First we show that $\mathcal{W}^0 > 0$. Consider the outcome $(\bar{q}, \bar{d}_k, k^*, \bar{e})$ given as follows: $\bar{q} = u^{-1}[(1+$

$r)k^*] > 0$, $\bar{e}$ solves

$$[\alpha(1/e) - \alpha'(1/e)/e][u(\bar{q}) - c(\bar{q})] = \psi'(e),$$

$\bar{d}_k = u(\bar{q}) - \psi'(\bar{e})/\alpha(1/\bar{e}) > c(\bar{q})$. The outcome is implementable and is associated with positive welfare.

Second, we show that $\mathcal{W}^c$ is achievable, that is, a constrained-efficient outcome (under the additional constraint $z = 0$), (q^c, d_k^c, k^c, e^c), exists. Note first that any outcome (q, d_k, k, e) with $q > q^*$ is strictly dominated by another outcome with $q' \leq q^*$; the proof follows exactly the same arguments as in the proof of Lemma 1.

Moreover, any outcome (q, d_k, k, e) with $d_k < k$ is strictly dominated as well. If $k > k^*$, then we can simply decline k and obtain higher welfare. Otherwise, assume that $k = k^*$ and consider two cases: (i) $q < q^*$. Then, consider another outcome (q', d_k', k, e) such that $q < q' = q + \varepsilon < q^*$ and that

$$u(q') - F'(k)d_k' = u(q) - F'(k)d_k.$$

Then,

$$F'(k)d_k' - c(q') = u(q') - c(q') - u(q) + F'(k)d_k \geq c(q) + F'(k)d_k.$$

The last inequality follows from the fact that $u(q') - c(q') > u(q) - c(q)$. So (q', d_k', k, e) is implementable but has strictly higher welfare. (ii) $q = q^*$ and $k = k^*$. Then, because $(1 + r)k^* < u(q^*) - \frac{\psi'(e^*)}{\alpha(1/e^*)}$ and because (q, d_k, k, e) satisfies (29), we have $e > e^*$. Consider (q, d_k', k, e') with $d_k' = d_k + \varepsilon < \min\{k, u(q)/(1 + r)\}$ and that

$$\frac{\psi'(e')}{\alpha(1/e')} = [u(q) - F'(k)d_k']$$

with $e' > e^*$. Then, (q, d_k', k, e') is implementable but has strictly higher welfare.

Thus, we may only consider outcomes with $k = d_k$, and $q \leq q^*$. This implies that $k \leq \hat{k}$ that is given by

$$F'(\hat{k})\hat{k} = u(q^*). \tag{71}$$

Therefore, we may consider outcomes of the form (q, k, k, e) that satisfies (26)-(29) and $q \in [0, q^*]$, $k \in [0, \hat{k}]$. Thus, we have a maximization problem of a continuous objective function with a compact feasible set, which admits a maximum.

Now we show that, in any constrained-efficient outcome, (q^c, d_k^c, k^c, e^c), $k^c > k^*$. Suppose, by contradiction, that $k^c = k^*$. Consider two cases.

(a) $q^c < q^*$. We have shown that $d_k^c = k^*$. Note that because $k^c = k^*$ and because (29) holds with equality, (26) holds with strict inequality. Let $k' > k^*$ be sufficiently close to k^* such that

(i) set q' as $u(q') - F'(k')k' = u(q^c) - F'(k^*)k^*$; $q' < q^*$;

(ii) $e^c\alpha(1/e^c)[u'(q') - c'(q')]\frac{g'(k')}{u'(q^c)} > [1 + r - F'(k')]$.

Here $g(k) = F'(k)k$ and, by assumption, g is concave. The second requirement can be satisfied because $1+r-F'(k^*) = 0$ but the left-hand side is bounded away from zero. Because $u'(q')-c'(q') > u(q^c) - c(q^c)$, it follows that $F'(k')k' > c(q')$. Thus, (q', k', k', e^c) is implementable but the welfare difference is ($\sigma = e^c\alpha(1/e^c)$)

$$\sigma[u(q') - c(q') - u(q^c) + c(q^c)] - \{[(1 + r)k' - F(k')] - [(1 + r)k^* - F(k^*)]\}$$

$$> \sigma[u'(q') - c'(q')]\frac{g'(k')}{u'(q^c)}[k' - k^*] - [(1 + r) - F'(k')][k' - k^*] > 0.$$

(b) $q^c = q^*$, and hence, $e^c > e^*$, for otherwise $(1+r)k^* \geq u(q^*) - \frac{\psi'(e^*)}{\alpha(1/e^*)}$. Let $k' > k^*$ be sufficiently close to k^* such that

(i) set e' as $u(q^*) - F'(k')k' = \psi'(e')/\alpha(1/e')$; $e^c > e' > e^*$;

(ii) let $j(e) = \psi'(e)/\alpha(1/e)$ and $l(e) = e\alpha(1/e)h(q^*) - \psi(e)$ (note that $j(e)$ is strictly increasing and hence $j'(e) > 0$), then k' is such that

$$l'(e')\frac{g'(k')}{\max_{e\in[e',e^0]} j'(e)} > [1 + r - F'(k')].$$

Again, the second requirement can be satisfied because $1 + r - F'(k^*) = 0$ but the left-hand side is bounded away from zero. (q^*, k', e') is implementable but the welfare difference is

$$[l(e') - l(e^c)] - \{[(1 + r)k' - F(k')] - [(1 + r)k^* - F(k^*)]\}$$

$$> l'(e')\frac{g'(k')}{\max_{e\in[e',e^c]} j'(e)}[k' - k^*] - [(1 + r) - F'(k')][k' - k^*] > 0.$$

Therefore, we have $k^c > k^*$.

Proof of Proposition 3

(1) The characterization and uniqueness of the first-best allocation as given by (18)-(20) follows similar arguments to those in Proposition 2 (1). Now we show that for all $i \in [0, i^{**}]$, the outcome $(q^*, d_z^*, k^*, z^*, k^*, e^*)$ with

$$d_z^* = u(q^*) - (1 + r)k^* - \frac{\psi'(e^*)}{\alpha(1/e^*)} > 0$$

satisfies constraints (12)-(16). Clearly, $F'(k^*) = 1+r$ implies (16) is satisfied. Note that (14) holds by construction. Plugging this into (12), it is straightforward to verify that it holds if and only if

$$i \leq i^{**} = \frac{e^*\psi'(e^*) - \psi(e^*)}{u(q^*) - (1+r)k^* - \psi'(e^*)/\alpha(1/e^*)}.$$

Note that (15) holds if and only if

$$u(q^*) - \psi'(e^*)/\alpha(1/e^*) \geq c(q^*),$$

that is,

$$\alpha(1/e^*)[u(q^*) - c(q^*)] \geq \psi'(e^*).$$

But, by (20) and the fact that $\alpha'(\theta) > 0$ for all θ,

$$\alpha(1/e^*)[u(q^*) - c(q^*)] = \alpha'(1/e^*)/e^*[u(q^*) - c(q^*)] + \psi'(e^*) > \psi'(e^*).$$

(2) First we show that when $i > i^{**}$, any outcome (q, d_z, d_k, z, k, e) with $d_z < z$ or $d_k < k$ is strictly dominated. Note that any outcome with $q > q^*$ is strictly dominated by another with $q' \leq q^*$. The case with $d_k < k$ follows the same arguments as those in the proof of Lemma 2. Consider the case with $d_z < z$. If $k > k^*$, then we may decrease k and d_k and increase d_z to keep the buyer surplus unchanged, and by doing so we keep the constraints but increase the welfare. So assume that $k = k^*$. If $q < q^*$, then we may increase both q and d_z to keep the buyer surplus unchanged, and by doing so doing so we keep the constraints but increase the welfare. So assume that $k = k^*$ and $q = q^*$. Then, by (12) and (14),

$$\frac{e\psi'(e) - \psi(e)}{i} \geq z > d_z = u(q^*) - \psi'(e)/\alpha(1/e) - (1+r)k^*,$$

and hence

$$\frac{e\psi'(e) - \psi(e)}{u(q^*) - \psi'(e)/\alpha(1/e) - (1+r)k^*} > i > i^{**},$$

which implies that $e > e^*$. Thus, we may increase d_z and decrease e to keep (14) intact, and by doing so we increase the welfare.

Thus, we may only consider outcomes with $d_k = k$, $d_z = z$, and with $q \leq q^*$. Because $q \leq q^*$, to satisfy (12) it must be the case that $F'(k)k \leq u(q^*)$, that is, $k \leq \hat{k}$, which is given by (71). Thus,

an outcome (q, z, k, e) is is implementable if and only if

$$-iz - [1 + r - F'(k)]k + e\alpha(1/e)[u(q) - z - F'(k)k] \geq \psi(e), \tag{72}$$

$$-c(q) + z + F'(k)k \geq 0, \tag{73}$$

$$1 + r \geq F'(k), \tag{74}$$

$$\psi'(e) = \alpha(1/e)[u(q) - z - F'(k)k]. \tag{75}$$

Note that because $d_z = z$ and $d_k = k$, $(q, d_z, d_k) \in \mathcal{CO}(z, k; R)$.

We begin with a claim.

Claim 1. (i) When $i = i^{**}$, the seller's participation constraint, (73), holds with strict inequality at the optimum, (q^*, z^*, k^*, e^*); (ii) for all $i > i^{**}$, the buyer's participation constraint, (72), binds, and $q^p < q^*$ at the optimum.

Proof. **(i)** We have shown it in (1).

(ii) To show that (72) binds for all $i > i^{**}$, we consider two cases:

(a) At the optimum, $k^p > k^*$. Suppose, by contradiction, that (72) does not bind. Let (z', k') be such that $k^* \leq k' < k^p$ but $z' + F'(k')k' = z^p + F'(k^p)k^p$, and, by continuity, the tuple (q^p, z', k', e^p) also satisfies (72). Because $k' < k^p$, this leads to an increase in the welfare, a contradiction.

(b) At the optimum, $k^p = k^*$. Consider the Lagrangian associated with (72), (73), (74), (75), $q \geq 0$, $z \geq 0$, and $e \geq 0$:

$$\begin{aligned}
\mathcal{L}(q, z, k, e; \lambda, \mu, \xi, \eta, \nu_q, \nu_z, \nu_e) &= e\alpha(1/e)[u(q) - c(q)] + [F(k) - (1 + r)k] - \psi(e) \\
&+ \lambda\{-iz - [(1 + r) - F'(k)]k + e\alpha(1/e)[u(q) - z - F'(k)k] - \psi(e)\} \\
&+ \mu\{[F'(k)k + z - c(q)]\} + \xi[(1 + r) - F'(k)] \\
&+ \eta\{\psi'(e) - \alpha(1/e)[u(q) - z - F'(k)k]\} \\
&+ \nu_q q + \nu_z z + \nu_e e,
\end{aligned}$$

where $\lambda \geq 0$, $\mu \geq 0$, $\xi \geq 0$, η, $\nu_q \geq 0$, $\nu_z \geq 0$, and $\nu_e \geq 0$ are the Lagrange multipliers associated with (72), (73), (74), (75), $q \geq 0$, $z \geq 0$, and $e \geq 0$ respectively. From the Kuhn-Tucker Theorem, the following are the first-order necessary conditions with respect to q, z, e (with $k^p = k^*$) and the

complementary slackness conditions for $q \geq 0$, $z \geq 0$, and $e \geq 0$:

$$e^p\alpha(1/e^p)[u'(q^p) - c'(q^p)] + \lambda e^p\alpha(1/e^p)u'(q^p) - \mu c'(q^p) - \eta\alpha(1/e^p)u'(q^p) + \nu_q = 0, \quad (76)$$

$$\lambda[i + e^p\alpha(1/e^p)] = \mu + \nu_z + \eta\alpha(1/e^p), \quad (77)$$

$$\begin{aligned}&[\alpha(1/e^p) - \alpha'(1/e^p)/e^p][u(q^p) - c(q^p)] - \psi'(e^p)\\ &+\lambda\{[\alpha(1/e^p) - \alpha'(1/e^p)/e^p][u(q^p) - z^p - (1+r)k^*] - \psi'(e^p)\} + \nu_e\\ &= -\eta\{\psi''(e^p) + [\alpha'(1/e^p)/(e^p)^2][u(q^p) - z^p - (1+r)k^*]\},\end{aligned} \quad (78)$$

$$\nu_q q^p = 0, \quad (79)$$

$$\nu_z z^p = 0, \quad (80)$$

$$\nu_e e^p = 0. \quad (81)$$

In addition, (72) and (73) are not binding only if $\lambda = 0$ and $\mu = 0$, respectively.

To verify that (72) binds for all $i > i^{**}$, i.e. $\lambda > 0$, suppose by contradiction that $\lambda = 0$. Then (72) holds with strict inequality, and hence $q^p > 0$ and $e^p > 0$. Then from (73), $q^p > 0$, $e^p > 0$, and $k^p = k^*$, we have $z^p > 0$. Consequently, $\nu_q = 0$, $\nu_z = 0$, and $\nu_e = 0$. Combining (76) and (77) with $\nu_q = 0$ and $\nu_z = 0$ yield

$$\frac{u'(q^p)}{c'(q^p)} = \frac{e^p\alpha(1/e^p) + \mu}{e^p\alpha(1/e^p) + \mu - \lambda i}. \quad (82)$$

From (82), $q^p = q^*$, and hence, from (77) and $\lambda = 0$ we have $-\alpha(1/e^p)\eta = \mu$. If $\mu = 0$, then $\eta = 0$ and from (78) and $\nu_e = 0$, we have $e^p = e^*$, a contradiction. If $\mu > 0$, then (73) is binding and hence $d_z^p + (1+r)k^* = c(q^*)$. By (77) and $\nu_z = 0$, we have $\mu = -\eta\alpha(1/e^p) > 0$. But then, by (78), this implies

$$[\alpha(1/e^p) - \alpha'(1/e^p)/e^p][u(q^*) - c(q^*)] - \psi'(e^p) > 0,$$

and hence $e^p < e^*$. However, by (75),

$$\psi'(e^p) = \alpha(1/e^p)[u(q^*) - c(q^*)],$$

and, because $c(q^*) < d_z^* + (1+r)k^*$, this implies that $e^p > e^*$. This leads to a contradiction. Hence $\lambda > 0$ and so (72) is binding. Moreover, because $\lambda > 0$, (82) implies that $u'(q^p) > c'(q^p)$ and hence $q^p < q^*$. □

Given that (72) and (75) bind, we can solve for z and q as a function of (k, e, i):

$$z(k, e, i) = \frac{1}{i}\{e\psi'(e) - \psi(e) - [1 + r - F'(k)]k\},$$

$$q(k,e,i) = f\left\{g(e,i) + \frac{[-(1+r)+(1+i)F'(k)]k}{i}\right\},$$

where

$$g(e,i) = \frac{1}{i}[e\psi'(e) - \psi(e)] + \frac{\psi'(e)}{\alpha(1/e)}$$

and $f(x) = u^{-1}(x)$. Then, the objective function can be written as

$$G(k,e,i) = e\alpha(1/e)\{u(q(k,e,i)) - c(q(k,e,i))\} - \psi(e) + F(k) - (1+r)k. \tag{83}$$

Claim 2. There exists an $\bar{i}'$ such that for all $i \in [i^{**}, \bar{i}']$, there is a unique maximizer $(k^p(i), e^p(i))$ for

$$\max_{k\in[k^*,\hat{k}],e\in[0,1]} G(k,e,i),$$

and $z^p(i) > 0$ for all such i's. Moreover, if $1 + r + F''(k^*)k^* < \frac{-F''(k^*)k^*}{i^{**}}$, then $\frac{d}{de}e^p(i) > 0$ for all $i \in [i^{**}, \bar{i}']$.

Proof. The proof applies the Implicit Function Theorem (IFT).

First we show that

$$\frac{\partial}{\partial e}G(k^*,e^*,i^{**}) = \frac{\partial}{\partial k}G(k^*,e^*,i^{**}) = 0. \tag{84}$$

The first partial derivative of $G(k,e,i)$ with respect to e at (k^*,e^*,i^{**}) is

$$\begin{aligned}
\frac{\partial G(k^*,e^*,i^{**})}{\partial e} &= \left\{\alpha(1/e^*) - \frac{\alpha'(1/e^*)}{e^*}\right\}\{u[q(k^*,e^*,i^{**})] - c[q(k^*,e^*,i^{**})]\} && (85)\\
&+ e^*\alpha(1/e^*)\frac{\partial q(k^*,e^*,i^{**})}{\partial e}\left\{u'[q(k^*,e^*,i^{**})] - c'[q(k^*,e^*,i^{**})]\right\} - \psi'(e^*) && (86)\\
&= \left[\alpha(1/e^*) - \frac{\alpha'(1/e^*)}{e^*}\right][u(q^*) - c(q^*)] - \psi'(e^*) = 0. && (87)
\end{aligned}$$

The first partial derivative of $G(k,e,i)$ with respect to k at (k^*,e^*,i^{**}) is

$$\begin{aligned}
&\frac{\partial G(k^*,e^*,i^{**})}{\partial k} && (88)\\
= \; & e^*\alpha(1/e^*)\left\{u'[q(k^*,e^*,i^{**})] - c'[q(k^*,e^*,i^{**})]\right\}\frac{\partial q(k^*,e^*,i^{**})}{\partial k} + F'(k^*) - (1+r) = 0. && (89)
\end{aligned}$$

Now we show that

$$\frac{\partial^2}{\partial k^2}G(k^*,e^*,i^{**})<0,\ \frac{\partial^2}{\partial e^2}G(k^*,e^*,i^{**})<0, \tag{90}$$
$$\frac{\partial^2}{\partial k^2}G(k^*,e^*,i^{**})\frac{\partial^2}{\partial e^2}G(k^*,e^*,i^{**})-\frac{\partial^2}{\partial k\partial e}G(k^*,e^*,i^{**})>0.$$

The second partial derivatives are

$$\begin{aligned}
&\frac{\partial^2 G(k^*,e^*,i^*)}{\partial e^2}=\alpha''(1/e^*)/(e^*)^3[u(q(k^*,e^*,i^{**}))-c(q(k^*,e^*,i^{**}))]\\
+\quad&[\alpha(1/e^*)-\alpha'(1/e^*)/e^*][u'(q(k^*,e^*,i^{**}))-c'(q(k^*,e^*,i^{**}))]\frac{\partial}{\partial e}q(k^*,e^*,i^{**})\\
+\quad&[u'(q(k^*,e^*,i^{**}))-c'(q(k^*,e^*,i^{**}))]\frac{\partial}{\partial e}|_{(k,e,i)=(k^*,e^*,i^{**})}\left\{e\alpha(1/e)\frac{\partial}{\partial e}q(k,e,i)\right\}\\
+\quad&e^*\alpha(1/e^*)\left[\frac{\partial}{\partial e}q(k^*,e^*,i^{**})\right]^2[u''(q(k^*,e^*,i^{**}))-c''(q(k^*,e^*,i^{**}))]-\psi''(e^*)\\
=\quad&\alpha''(1/e^*)/(e^*)^3[u(q^*)-c(q^*)]+e^*\alpha(1/e^*)\left[\frac{\partial}{\partial e}q(k^*,e^*,i^{**})\right]^2[u''(q^*)-c''(q^*)]-\psi''(e^*)<0,
\end{aligned}$$

$$\begin{aligned}
&\frac{\partial^2 G(k^*,e^*,i^*)}{\partial k^2}=e^*\alpha(1/e^*)\left[\frac{\partial}{\partial k}q(k^*,e^*,i^{**})\right]^2[u''(q(k^*,e^*,i^{**}))-c''(q(k^*,e^*,i^{**}))]\\
+\quad&e^*\alpha(1/e^*)[u'(q(k^*,e^*,i^{**}))-c'(q(k^*,e^*,i^{**}))]\frac{\partial^2}{\partial k^2}q(k^*,e^*,i^{**})+F''(k^*)\\
=\quad&e^*\alpha(1/e^*)\left[\frac{\partial}{\partial k}q(k^*,e^*,i^{**})\right]^2[u''(q^*)-c''(q^*)]+F''(k^*)<0,
\end{aligned}$$

and

$$\begin{aligned}
&\frac{\partial^2 G(k^*,e^*,i^*)}{\partial k\partial e}=[u'(q(k^*,e^*,i^{**}))-c'(q(k^*,e^*,i^{**}))]\frac{\partial}{\partial e}|_{(k,e,i)=(k^*,e^*,i^{**})}\left\{e\alpha(1/e)\frac{\partial}{\partial k}q(k,e,i)\right\}\\
+\quad&e^*\alpha(1/e^*)\left[\frac{\partial}{\partial k}q(k^*,e^*,i^{**})\frac{\partial}{\partial e}q(k^*,e^*,i^{**})\right][u''(q(k^*,e^*,i^{**}))-c''(q(k^*,e^*,i^{**}))]\\
=\quad&e^*\alpha(1/e^*)\left[\frac{\partial}{\partial k}q(k^*,e^*,i^{**})\frac{\partial}{\partial e}q(k^*,e^*,i^{**})\right][u''(q^*)-c''(q^*)].
\end{aligned}$$

Now,

$$\begin{aligned}
& \frac{\partial^2 G(k^*,e^*,i^*)}{\partial e^2}\frac{\partial^2 G(k^*,e^*,i^*)}{\partial k^2} \\
> \; & e^*\alpha(1/e^*)\left[\frac{\partial}{\partial e}q(k^*,e^*,i^{**})\right]^2[u''(q^*)-c''(q^*)]e^*\alpha(1/e^*)\left[\frac{\partial}{\partial k}q(k^*,e^*,i^{**})\right]^2[u''(q^*)-c''(q^*)] \\
= \; & \left\{\frac{\partial^2 G(k^*,e^*,i^*)}{\partial k\partial e}\right\}^2.
\end{aligned}$$

Because of (90), (85), and (88), and by the IFT, there is an open neighborhood $O=(k_0,k_1)\times(e_0,e_1)\times(i_0,i_1)$ around (e^*,i^*) and a continuously differentiable implicit function $(k^p,e_0^p):(i_0,i_1)\to(k_0,k_1)\times(e_0,e_1)$ such that for all $i\in[i^{**},i_1)$, $[k^p(i),e_0^p(i)]$ is the unique $(k,e)\in(k_0,k_1)\times(e_0,e_1)$ such that

$$\frac{\partial}{\partial e}G(k^p(i),e_0^p(i),i)=0 \text{ and } \frac{\partial}{\partial k}G(k^p(i),e_0^p(i),i)=0,$$

and another continuously differentiable implicit function $e^p:(i_0,i_1)\to(e_0,e_1)$ such that for all $i\in[i^{**},i_1)$, $e^p(i)$ is the unique $e\in(e_0,e_1)$ such that

$$\frac{\partial}{\partial e}G(k^*,e^p(i),i)=0,$$

and that (90) holds over O. Suppose first that the constraint $k\geq k^*$ is not binding at the optimum and hence, by the Kuhn-Tucker conditions, $k^p(i)>k^*$. Using the same arguments as those in Proposition 2, we can show that $(k^p(i),e_0^p(i))$ is the global maximizer as well and hence

$$(q^p,z^p,k^p,e^p)=(q[k^p(i),e_0^p(i),i],z[k^p(i),e_0^p(i),i],k^p(i),e_0^p(i))$$

is the unique constrained-efficient outcome. Note that, by continuity, $e_0^p(i)>0$ and $k^p(i)$ is close to k^* at least locally and hence $z^p>0$. Alternatively, the constraint $k\geq k^*$ is binding and hence $(k^*,e^p(i))$ is the unique solution, and $z^p>0$ at least locally as well.

Now we assume $1+r+F''(k^*)k^*<\frac{-F''(k^*)k^*}{i^{**}}$ and show that at the optimum, $k^p=k^*$ and $e^p(i)$ is increasing. For all $i\in[i^{**},i_1)$, let $q(i)=q(k^*,e^p(i),i)$,

$$\begin{aligned}
& \frac{\partial}{\partial k}G(k^*,e^p(i),i) \\
= \; & e^p(i)\alpha(1/e^p(i))[u'(q(k^*,e^p(i),i))-c'(q(k^*,e^p(i),i))]\frac{\partial}{\partial k}q(k^*,e^p(i),i)+F'(k^*)-(1+r) \\
= \; & e^p(i)\alpha(1/e^p(i))[u'(q(i))-c'(q(i))]f'[u(q(i))]\left\{[1+r+F''(k^*)k^*]+F''(k^*)k^*/i\right\}.
\end{aligned}$$

Note that using the same arguments as in the proof of Proposition 2, we can show that

$$(q(k^*, e^p(i), i), d(k^*, e^p(i), i), e^p(i))$$

is a constrained-efficient outcome under the restriction that $k = k^*$. Hence, if $q(k^*, e^p(i), i) > q^*$, then we may use a similar argument as that in Proposition 2 to derive a contradiction. Thus, $q(k^*, e^p(i), i) \leq q^*$.

Because

$$1 + r + F''(k^*)k^* < \frac{-F''(k^*)k^*}{i^{**}},$$

there exists $i_2 < i_1$ such that for all $i \in [i^{**}, i_2]$,

$$1 + r + F''(k^*)k^* + \frac{F''(k^*)k^*}{i} \leq 0,$$

and hence, for all such i's,

$$\frac{\partial}{\partial k} G(k^*, e^p(i), i) \leq 0.$$

Recall that (90) holds over O, and hence $G(k, e, i)$ is strictly concave over O. Because

$$\frac{\partial}{\partial e} G(k^*, e^p(i), i) = 0 \text{ and } \frac{\partial}{\partial k} G(k^*, e^p(i), i) \leq 0$$

for all $i \in [i^{**}, i_2]$, it follows that $e^p(i)$ is the local maximizer in the neighborhood $(k_0, k_1) \times (e_0, e_1)$ for all $i \in [i^{**}, i_2]$.

To show that it is also the global maximizer, first consider

$$M(i) = \max\{G(k, e, i) : (k, e) \in [k^*, \hat{k}] \times [0, 1] - (k_0, k_1) \times (e_0, e_1)\}.$$

By the Theorem of the Maximum in e.g. Stokey and Lucas (1989), $M(i)$ is continuous and $M(i^{**}) < G(k^*, e^*, i^{**})$. Let $\delta = G(k^*, e^*, i^{**}) - M(i^{**}) > 0$. Then by continuity, there exists an $i_3 \in (i^{**}, i_2]$ such that if $i \in [i^{**}, i_3]$, then $M(i) \leq M(i^{**}) + \delta/3 < G(k^*, e^*, i^{**}) - \delta/3 \leq G(k^*, e^p(i), i)$. Hence, for all $i \in [i^*, i_3]$, $(k^*, e^p(i))$ maximizes $G(\cdot, \cdot, i)$ subject to $k \geq k^*$. Moreover, because the function $-c(q(k^*, e, i)) + d_z(k^*, e, i) + (1 + r)k^*$ is continuous and because $-c(q^*) + d_z^* + (1 + r)k^* \geq 0$, it follows from continuity that there exists an $i_4 \in (i^{**}, i_3]$ such that for all $i \in (i^{**}, i_4]$, $-c(q(k^*, e^p(i), i)) + d_z(k^*, e^p(i), i) + (1 + r)k^* \geq 0$. Thus, for each $i \in (i^{**}, i_4]$, $(q^p, d_z^p, k^p, e^p) = (q(k^*, e^p(i), i), d_z(k^*, e^p(i), i), k^*, e^p(i))$ is the constrained-efficient outcome.

Finally, by IFT again, $e^p(i)$ is continuously differentiable and for all $i \in (i^*, i_4]$,

$$(e^p)'(i) = -\frac{\partial^2}{\partial e \partial i} G(k^*, e^p(i), i) / \frac{\partial^2}{\partial e^2} G(k^*, e^p(i), i).$$

We have shown that $\frac{\partial^2}{\partial e^2} G(k^*, e^*, i^{**}) < 0$. Here we show that

$$\frac{\partial^2}{\partial e \partial i} G(k^*, e^*, i^{**}) > 0. \tag{91}$$

To compute (91), we differentiate to obtain

$$\begin{aligned}
& \frac{\partial^2}{\partial e \partial i} G(k^*, e^*, i^{**}) = [\alpha(1/e^*) - \alpha'(1/e^*)/e^*][u'(q^*) - c'(q^*)] \frac{\partial}{\partial i} q(k^*, e^*, i^{**}) \\
+ \quad & e^* \alpha(1/e^*)[u'(q^*) - c'(q^*)] \frac{\partial^2}{\partial e \partial i} q(k^*, e^*, i^{**}) \\
+ \quad & e^* \alpha(1/e^*)[u''(q^*) - c''(q^*)] \frac{\partial}{\partial e} q(k^*, e^*, i^{**}) \frac{\partial}{\partial i} q(k^*, e^*, i^{**}) \\
= \quad & e^* \alpha(1/e^*)[u''(q^*) - c''(q^*)] \frac{\partial}{\partial e} q(k^*, e^*, i^{**}) \frac{\partial}{\partial i} q(k^*, e^*, i^{**}) \\
= \quad & e^* \alpha(1/e^*)[u''(q^*) - c''(q^*)][f'(u(q^*))]^2 g_e(e^*, i^{**}) g_i(e^*, i^{**}).
\end{aligned}$$

Now,

$$g_e(e^*, i^{**}) = \frac{e^* \psi''(e^*)}{i^{**}} + \frac{\psi''(e^*)\alpha(1/e^*) + \psi'(e^*)\alpha'(1/e^*)/(e^*)^2}{\alpha(1/e^*)^2} > 0,$$

and

$$g_i(e^*, i^{**}) = \frac{e^* \psi'(e^*) - \psi(e^*)}{(i^{**})^2} < 0,$$

and hence $\frac{\partial^2}{\partial e \partial i} G(k^*, e^*, i^{**}) > 0$. By continuity, there exists $i_5 \leq i_4$ and $e_0' \geq e_0$ and $e_2 \leq e_1$ such that for all $(e, i) \in [e_0', e_2] \times [i^{**}, i_5]$,

$$\frac{\partial^2}{\partial e \partial i} G(k^*, e, i) > 0 \text{ and } \frac{\partial^2}{\partial e \partial i} G(k^*, e, i) > 0.$$

Let $\bar{i}' \leq i_5$ be such that $e^p(i) \in (e_0', e_2)$ for all $i \in [i^{**}, \bar{i}']$. Hence, for $i \in (i^*, \bar{i}']$, $\frac{d}{di} e^p(i) > 0$ and hence $e^p(i) > e^*$. □

(3) Recall from Lemma 1 that for any i and in any constrained-efficient outcome w.r.t. i, $e^p(i) \leq \hat{e}$.

Moreover, by (72), we have

$$z^p(i) \leq e^p(i)\alpha(1/e^p(i))[u(q^p(i)) - c(q^p(i))]/i \leq \hat{e}\alpha(1/\hat{e})[u(q^*) - c(q^*)]/i.$$

Claim 1. Denote by $\mathcal{W}(i)$ the welfare associated with a constrained-efficient outcome. For any $\varepsilon > 0$, there exists i_ε for which $i > i_\varepsilon$ implies $\mathcal{W}(i) \leq \mathcal{W}^k + \varepsilon$.

Proof. First note that in any constrained-efficient outcome, $q^p(i) \leq q^*$ and $k^p(i) \leq \hat{k}$. For each i, define $\tilde{k}(i)$ by the capital stock that satisfies

$$F'(\tilde{k}(i))\tilde{k}(i) - F'(\hat{k})\hat{k} = \frac{\hat{e}\alpha(1/\hat{e})[u(q^*) - c(q^*)]}{i}.$$

Because the function $F'(k)k$ is strictly increasing in k with range $\mathbb{R}_+$, $\tilde{k}(i)$ is well-defined and is a decreasing function of i. Moreover, as $i \to \infty$, $\tilde{k}(i)$ converges to $\hat{k}$.

Let $S(k) = F'(k)k$. Given $\epsilon > 0$, let i_ϵ be so large that $i > i_\epsilon$ implies

$$\{1 + r - F'[\tilde{k}(i)]\}[\tilde{k}(i) - \hat{k}] \leq \epsilon, \tag{92}$$

$$S'(\tilde{k}(i))(1 + i) \geq 1 + r. \tag{93}$$

Note that i_ϵ is well-defined because $\tilde{k}(i)$ converges to $\hat{k}$ and S' is a decreasing function.

Now we show that if $i > i_\epsilon$, then $\mathcal{W}(i) \leq \mathcal{W}^k + \epsilon$.

Fix some $i > i_\epsilon$, and let $(q^p(i), d_z^p(i), d_k^p(i), z^p(i), k^p(i), e^p(i))$ be a constrained-efficient outcome. Consider an alternative outcome

$$(q', d_z', d_k', z', k', e') = (q^p(i), 0, d_k', 0, k', e^p(i)),$$

where k' and d_k' are such that

$$F'(k')k' - F'[k^p(i)]k^p(i) = z^p(i) \leq \frac{\hat{e}\alpha(1/\hat{e})[u(q^*) - c(q^*)]}{i}, \tag{94}$$

$$F'(k')d_k' = F'[k^p(i)]d_k^p(i) + d_z^p(i). \tag{95}$$

Note that $k' \leq \tilde{k}(i)$. Now we show that the outcome $(q', d_z', d_k', z', k', e')$ satisfies incentive compatibility constraints (72)-(75) and has welfare equal to $\mathcal{W}' \geq \mathcal{W}(i) - \epsilon$. Note that, by definition, $\mathcal{W}' \leq \mathcal{W}^k$ and hence this implies that $\mathcal{W}^k \geq \mathcal{W}(i) - \epsilon$.

First consider the buyer's participation constraint, (72). Because the original outcome satisfies

(72), it suffices to show that

$$-iz^p(i) - [1 + r - F'(k^p(i))]k^p(i) \leq -[1 + r - F'(k')]k',$$

which holds if and only if

$$\begin{aligned} & (1+r)(k' - k^p(i)) - z^p(i) \leq iz^p(i) \\ \Leftrightarrow \quad & (1+r)(k' - k^p(i)) \leq (1+i)z^p(i) \\ \Leftrightarrow \quad & \frac{z^p(i)}{k' - k^p(i)} \geq \frac{1+r}{1+i}. \end{aligned}$$

By definition of k', $z^p(i) = F'(k')k' - F'(k^p(i))k^p(i)$ and hence (note that $k' \leq \tilde{k}(i)$), by (93),

$$\begin{aligned} \frac{z^p(i)}{k' - k^p(i)} &= \frac{F'(k')k' - F'(k^p(i))k^p(i)}{k' - k^p(i)} \\ &\geq S'(k') \geq S'(\tilde{k}(i)) \\ &\geq \frac{1+r}{1+i}. \end{aligned}$$

In addition, because $k' \geq k^p(i)$ and because of (95), the alternative outcome satisfies (73)–(75).

Here we show that $\mathcal{W}' \geq \mathcal{W}(i) - \epsilon$. First note that

$$\begin{aligned} [F(k^p(i)) - (1+r)k^p(i)] - [F(k') - (1+r)k'] &\leq [F'(k') - (1+r)][k^p(i) - k'] \\ &= [1 + r - F'(k')][k' - k^p(i)]. \end{aligned}$$

Then, note that, in terms of variables relevant to the welfare, the alternative outcome differ from the original outcome only in the capital stock, and hence the difference in welfare, $\mathcal{W}' - \mathcal{W}(i)$, can be written as

$$\begin{aligned} \mathcal{W}' - \mathcal{W}(i) &= -\{[F(k^p(i)) - (1+r)k^p(i)] - [F'(k') - (1+r)k']\} \\ &\geq -[1 + r - F'(k')][k' - k^p(i)] \\ &\geq -[1 + r - F'(\tilde{k}(i))][\tilde{k}(i) - \hat{k}] \geq -\epsilon. \end{aligned}$$

The second last inequality follows from the fact that $k' - k^p(i) = z^p(i) \geq \tilde{k}(i) - \hat{k}$ and the fact that the function $S(k) = F'(k)k$ is concave in k, and the last inequality follows from (92). Hence, $\mathcal{W}' \geq \mathcal{W}(i) - \epsilon$. □

Because it is always feasible to set $z = 0$ and hence $\mathcal{W}(i) \geq \mathcal{W}^k$ for all i, the result that $\lim_{i\to\infty} \mathcal{W}(i) = \mathcal{W}^k$ follows immediately from Claim 1.

By Lemma 2, if we impose the additional constraints $z = 0$ and $k = k^*$, then the resulting maximum welfare, denoted $\mathcal{W}^0$, is strictly less than $\mathcal{W}^k$. let $[\mathcal{W}^k - \mathcal{W}^0]/2 > 0$. The following claim shows that, if we impose $k = k^*$, then, for i sufficiently large, the maximum achievable welfare is less than $\mathcal{W}^k - [\mathcal{W}^k - \mathcal{W}^0]/2$.

Claim 2. Define $\mathcal{W}^0(i)$ to be the maximum welfare achievable by outcomes satisfying $k = k^*$, together with constraints (72)-(75). There exists an $\hat{i}$ such that for all $i > \hat{i}$, $\mathcal{W}^0(i) < \mathcal{W}^k - [\mathcal{W}^k - \mathcal{W}^0]/2$.

Proof. We show that for any $\epsilon > 0$, there exists i'_ϵ such that $\mathcal{W}^0(i) < \mathcal{W}^0 + \epsilon$ for all $i > i'_\epsilon$. The claim follows immediately.

First note that since $\mathcal{W}^0(i) > 0$ (as it is always feasible to set $k = k^*$, q be such that $c(q) = (1+r)k^*$, and e solve $\psi'(e)/\alpha(1/e) = [u(q) - (1+r)k^*] > 0$) for all i, we can find a lower bound $\underline{q}$ and $\underline{e}$ such that for any outcome $(q^0(i), z^0(i), k^0(i), e^0(i))$ that achieves the maximum welfare under the constraints (72)-(75) and $k = k^*$, we have $q^0(i) > \underline{q}$ and $e^0(i) > \underline{e}$ for all i. Note that as $i > i^{**}$, at the optimum we must have $d^0_z(i) = z^0(i)$ and $d^0_k(i) = k^*$. Moreover, it follows that we can choose $\underline{q}$ to be strictly greater than $(1+r)k^*$, for otherwise the buyer will have arbitrarily small surplus and hence the search intensity will be arbitrarily small as well.

Now, the welfare, as a function of (q, k, e), is continuous and hence is uniformly continuous in $[\underline{q}, q^*] \times \{k^*\} \times [\underline{e}, \hat{e}]$. Thus, there exists $\delta > 0$ such that if $\|(q, e) - (q^0(i), e^0(i))\| < \delta$, then the welfare associated with (q, k^*, e), differs from the welfare $\mathcal{W}^0(i)$ by less than ϵ for all i.

Let $l(e) = \psi'(e)/\alpha(1/e)$. Then, $l'(e) > 0$ for all $e \in [\underline{e}, \hat{e}]$ and hence $A = \min_{e\in[\underline{e},\hat{e}]} l'(e) > 0$. Let i_ϵ be so large that if $i > i_\epsilon$,

$$\max\{2, 1 + u'(\underline{q})/A\} \frac{[u(q^*) - c(q^*)]}{c'(\underline{q}/2)i} < \min\{\underline{q}/2, \delta/2, \underline{q} - u^{-1}[(1+r)k^*]\}, \tag{96}$$

Fix an $i > i_\epsilon$ and an outcome $(q^0(i), z^0(i), k^*, e^0(i))$ that achieves $\mathcal{W}^0(i)$. We construct an alternative outcome, $(q', 0, k^*, e')$ such that $\|(q', e') - (q^0(i), e^0(i))\| < \delta$ and satisfies (72)-(75). Then, the welfare associated with the alternative outcome, denoted by $\mathcal{W}'$, is within ϵ of $\mathcal{W}^0(i)$, but $\mathcal{W}' \leq \mathcal{W}^0$.

The outcome $(q', 0, k^*, e')$ is given by

$$c(q') = c(q^0(i)) - z^0(i) \geq 0$$

and

$$\frac{\psi'(e')}{\alpha(1/e')} = u(q') - (1+r)k^*.$$

Because $z^0(i) \leq [u(q^*) - c(q^*)]/i$, it follows from (96) that $q' \geq \underline{q}/2$ and that $u(q') \geq (1+r)q^*$. Moreover, because $(q^0(i), z^0(i), k^*, e^0(i))$ satisfies (73),

$$-c(q') + (1+r)k^* = -c(q^0(i)) + z^0(i) + (1+r)k^* \geq 0,$$

and hence $(q', 0, k^*, e')$ satisfies (73) as well. Note that it also satisfies (72) and (75) by construction.

Thus, we have

$$0 \leq c(q^0(i)) - c(q') \leq z^0(i) \leq \frac{\hat{e}\alpha(1/\hat{e})[u(q^*) - c(q^*)]}{i},$$

and so, by (96),

$$|q^o(i) - q'| \leq |c(q^0(i)) - c(q')|/c'(\underline{q}/2) \leq \delta/2.$$

By (75),

$$|l(e^0(i)) - l(e')| = |\psi'(e^0(i))/\alpha(e^0(i)) - \psi'(e')/\alpha(1/e')| = |u(q^0(i)) - u(q') - z^0(i)| \leq u'(\underline{q}/2)[q^0(i) - q'] + z^0(i),$$

and so

$$|e' - e^0(i)| \leq (1/A)[u'(\underline{q}/2)[q^0(i) - q'] + z^0(i)] < \delta/2$$

by (96).

Thus, we have $\|(q', e') - (q^0(i), e^0(i))\| < \delta$, and hence $\mathcal{W}^0 \geq \mathcal{W}' > \mathcal{W}^0(i) - \epsilon$. Finally, take $\hat{i} = i'_{[\mathcal{W}^k - \mathcal{W}^0]/2}$. □

Now we show that for all $i > \hat{i}$, $k^p(i) > k^*$. Suppose not. Then $\mathcal{W}(i) = \mathcal{W}^0(i) < \mathcal{W}^k$, a contradiction.

Proof of Lemma 3

The necessity of those conditions are established in the main text. The sufficiency follows exactly the same arguments as those in the proof of Proposition 1. Note that as $n \to 1$, $\alpha(1/n) \to 0$, it follows that (35) cannot be satisfied at $n^p = 1$ with $v > 0$.

Proof of Lemma 4

If $i \leq i^*$, then, by Proposition 4, the first-best allocation is implementable with money alone. Moreover, as is shown in the proof of that proposition below, we may choose $d_z^p = z^p$ in the con-

strained efficient outcome and hence it solves the problem (40)-(42). The pairwise core requirement is obviously satisfied.

Clearly, for any solution, (q^p, k^p, n^p), to (40)-(42), the outcome

$$(q^p, d_z^p, d_k^p, z^p, k^p, n^p) = (q^p, z^p, k^p, z^p, k^p, n^p)$$

also satisfies (12)-(15). Here we show that, if $i > i^*$, then for any constrained efficient outcome, $(q^p, d_z^p, d_k^p, z^p, k^p, n^p)$, $d_z^p = z^p$, $k^p = d_k^p$, and (q^p, z^p, k^p, n^p) satisfies (40)-(42) and belongs to the set $[0, q^*] \times [0, \bar{z}] \times [0, \bar{k}] \times [0, n^*]$. Notice that if $q^p > q^*$, then we may decrease q^p and increase z^p so that (12) is unchanged but the welfare is increased. So $q^p \leq q^*$. Moreover, if $n^p > n^*$, then, because $q^p \leq q^*$, we may decrease n^p and increase z^p to keep (12) unchanged but the welfare is increased. This also implies the pairwise core requirement is satisfied.

Now suppose that $i > i^*$. Suppose that $(q^p, d_z^p, d_k^p, z^p, k^p, n^p)$ is an implementable outcome w.r.t. i. We have two claims.

(a) $k^p = d_k^p$. Suppose that $k^p > d_p^k$. Then we may decrease k^p (and increase z^p proportionally to keep (35) in tact) and increase $\mathcal{W}$, a contradiction.

(b) $z^p = d_z^p$.

(b.1) Suppose that $q^p < q^*$ and $z^p > d_z^p$. Let $d_z' = d_z^p + \epsilon < z^p$ be such that $u(q^p) + \epsilon \leq u(q^*)$. Let q' be such that $u(q') = u(q^p) + \epsilon$. Then,

$$-iz^p - (1+r-A)k^p + \alpha(1/n^p)[u(q') - d_z' - Ad_k^p] = -iz^p - (1+r-A)k^p + \alpha(1/n^p)[u(q^p) - d_z^p - Ad_k^p] = v,$$

and

$$-c(q') + d_z' + Ad_k^p = -[c(q') - c(q^p) - \epsilon] + [-c(q^p) + d_z^p + Ad_k^p]$$

$$\geq \epsilon - c'(q')(q' - q^p) \geq \epsilon - u'(q')(q' - q^p) \geq \epsilon - [u(q') - u(q^p)] = 0.$$

Thus, $(q^p, d_z', d_k^p, z^p, k^p, n^p)$ is implementable but has higher welfare as $q' > q^p$, a contradiction.

(b.2) Suppose that $q^p = q^*$ and $z^p > d_z^p$. If $n^p < n^*$, then we can decrease z^p and increase n^p to keep (35) unchanged but increase the welfare, a contradiction. Suppose that $n^p = n^*$. By Proposition 2, it must be the case that $k^p = d_k^p > 0$. Then we may increase d_z^p and decrease k^p to make $d_z' + Ak' = d_z^p + Ak^p$ while changing z^p so that (35) is unchanged, but the welfare is increased, a contradiction.

Proof of Proposition 4

First we consider a pure currency economy without capital. In that case, an outcome consists of (q^p, d_z^p, z^p, n^p). We have the following claim.

Claim 0. Consider an economy without capital, that is, with the additional restriction that $k = 0$. There exists (d_z^p, z^p) such that (q^p, d_z^p, z^p, n^p) is a constrained-efficient outcome if the pair (q^p, n^p) solves

$$\max_{(q,n)} n\alpha(1/n)[u(q) - c(q)] - nv \tag{97}$$

subject to

$$\alpha(1/n)[u(q) - c(q)] \geq ic(q) + v. \tag{98}$$

Proof. Suppose that (q^p, d_z^p, z^p, n^p) is a constrained-efficient outcome and suppose that (q^0, n^0) solves (97).

First we prove that (q^p, n^p) solves (97). Note that by implementability,

$$-iz^p + \alpha(1/n^p)[u(q^p) - d_z^p] = v,$$
$$z^p \geq d_z^p, \ -c(q^p) + d_z^p \geq 0.$$

Now, from the first inequality and that $z^p \geq d_z^p$ we have

$$\alpha(1/n^p)u(q^p) - v \geq [i + \alpha(1/n^p)]d_z^p \geq [i + \alpha(1/n^p)]c(q^p),$$

and hence

$$\alpha(1/n^p)[u(q^p) - c(q^p)] \geq v + ic(q^p),$$

i.e., (q^p, n^p) satisfies (98). Now suppose that (q^0, n^0) gives a higher value than (q^p, n^p) to (97). Let

$$z^0 = d_z^0 = \frac{\alpha(1/n)u(q^0) - v}{i + \alpha(1/n)}.$$

Then, (q^0, d_z^0, z^0, n^0) is implementable (note that the pairwise core requirement is satisfied because $z^0 = d_p^0$), a contradiction to (q^p, d_z^p, z^p, n^p) being constrained efficient. So (q^p, n^p) solves (97).

Conversely, we show that (q^0, d_z^0, z^0, n^0) is constrained-efficient. Suppose that it is not. Because it is implementable, it follows that (q^p, d_z^p, z^p, n^p) gives a higher value to (97) than (q^0, d_z^0, z^0, n^0). But (q^p, n^p) satisfies (98) and this leads to a contradiction to the fact that (q^0, n^0) solves (97). □

(1) Let $i \in [0, i^*]$. Then, by Claim 0, to show that the first-best allocation, (q^*, n^*) and $k = 0$, is

implementable, it is sufficient to show that (q^*, n^*) satisfies (98), that is,

$$\alpha(1/n^*)[u(q^*) - c(q^*)] \geq ic(q^*) + v,$$

which is equivalent to

$$i \leq \frac{\alpha(1/n^*)[u(q^*) - c(q^*)] - v}{c(q^*)} = i^*.$$

(2) Suppose that $i > i^*$. We first impose the constraint $k = 0$ and then we show that the constraint is binding.

Consider the economy without capital, that is, with the additional constraint $k = 0$. We consider the maximization problem (97) subject to (98) at equality.

Claim 1. Consider the economy without capital and suppose that $i > i^*$. Then, (98) binds at the optimum for the problem (97)-(98), and $q < q^*$ at the optimum.

Proof. Consider the Lagrangian associated with the maximization problem (97) subject to (98), $q \geq 0$, and $n \geq 0$:

$$\begin{aligned}
\mathcal{L}(q, n; \lambda, \nu_q, \nu_n) &= n\alpha\,(1/n)\,[u(q) - c(q)] - nv \\
&+ \lambda\{-ic(q) + \alpha(1/n)[u(q) - c(q)] - v\} \\
&+ \nu_q q + \nu_n n,
\end{aligned}$$

where $\lambda \geq 0$, $\nu_q \geq 0$, and $\nu_n \geq 0$ are the Lagrange multipliers associated with (98), $q \geq 0$, and $n \geq 0$ respectively. From the Kuhn-Tucker Theorem, the first-order necessary conditions with respect to q, n and the complementary slackness conditions for $q \geq 0$ and $n \geq 0$ are respectively

$$[\alpha(1/n^p)(n^p + \lambda)][u'(q^p) - c'(q^p)] - \lambda i c'(q^p) + \nu_q = 0 \tag{99}$$

$$\left[\alpha(1/n^p) - \alpha'(1/n^p)\left(\frac{1}{n^p} + \frac{\lambda}{(n^p)^2}\right)\right][u(q^p) - c(q^p)] + \nu_n = v \tag{100}$$

$$\nu_q q^p = 0 \tag{101}$$

$$\nu_n n^p = 0. \tag{102}$$

To show that (98) binds for $i > i^*$; i.e. $\lambda > 0$, suppose by contradiction that $\lambda = 0$. Given $v > 0$ and (98) holds with strict inequality, $q^p > 0$ and $n^p > 0$. Then from (101) and (102), $\nu_q = 0$ and $\nu_n = 0$. Hence from (99), (100), $\nu_q = 0$, and $\nu_n = 0$, $q^p = q^*$ and $n^p = n^*$. Consequently,

$$i \leq \frac{\alpha(1/n^*)[u(q^*) - c(q^*)] - v}{c(q^*)} = i^*$$

Hence, $i \leq i^*$, a contradiction. Thus for $i > i^*$, $\lambda > 0$ and hence (42) binds.

To verify that $q^p < q^*$ for all $i > i^*$, first note that from (99), $q^p \neq q^*$ unless $\lambda = 0$ and $\nu_q = 0$ which is violated when $i > i^*$. Now suppose $q^p > q^*$ and consider a deviation that decreases q^p to q^*, which still satisfies (42) . This produces higher welfare and is incentive feasible, a contradiction. Hence $q^p < q^*$.

□

The problem with (98) at equality simplifies to a choice of n. That is, the optimal trading mechanism solves (97) subject to

$$\alpha\,(1/n)\,[u(q) - c(q)] = ic(q) + v.$$

Let $q = g(n, i) > 0$ solve

$$\alpha\left(\frac{1}{n}\right)[u(g(n,i)) - c(g(n,i))] = ic(g(n,i)) + v.$$

Substituting q by the function g, we may rewrite the objective function (97) as

$$n\alpha(1/n)\{u[g(n,i)] - c[g(n,i)]\} - nv = n \cdot i \cdot c[g(n,i)].$$

Thus, the problem to maximize (97) subject to (98) at equality can be reduced to

$$\max_{n \in [0, n^*]} n \cdot i \cdot c[g(n,i)]. \tag{103}$$

Claim 2. Consider the economy without capital and consider the corresponding maximization problem (103). There exists $\bar{i}' > i^*$ such that for each $i \in (i^*, \bar{i}']$, there exists a unique $n^p(i)$ that solves its associated F.O.C., and that is the global maximizer of the problem. Moreover, the outcome $(q^p(i), n^p(i)) = [g(n^p(i), i), n^p(i)]$ is the unique constrained-efficient outcome with $\frac{d}{di} n^p(i) < 0$.

Proof. Fix some i. Define

$$f(n,i) \equiv i\{c[g(n,i)] + nc'[g(n,i)]g_n(n,i)\},$$

that is, $f(n,i) = \frac{\partial}{\partial n}\{n \cdot i \cdot c[g(n,i)]\}$. We apply the Implicit Function Theorem (IFT) in the neighborhood of the first-best to obtain a solution to $f(n,i) = 0$. To do so, we first determine the signs of the second derivatives, $f_n(n^*, i^*)$ and $f_i(n^*, i^*)$.

Let $h(q) = u(q) - c(q)$. Differentiating $\alpha(1/n)[h(g(n,i))] = ic(g(n,i)) + v$, we have

$$\begin{aligned}
g_n(n,i) &= \frac{\alpha'(1/n)h(q)}{n^2[\alpha(1/n)h'(q) - ic'(q)]};\\
g_i(n,i) &= \frac{c(q)}{\alpha(1/n)h'(q) - ic'(q)};\\
g_{ni}(n,i) &= \frac{c'(q)g_n(n,i)[\alpha(1/n)h'(q) - ic'(q)]}{[\alpha(1/n)h'(q) - ic'(q)]^2}\\
&- \frac{c(q)[\alpha(1/n)h''(q)g_n(n,i) - (1/n^2)\alpha'(1/n)h'(q) - ic''(q)g_n(n,i)]}{[\alpha(1/n)h'(q) - ic'(q)]^2};\\
g_{nn}(n,i) &= \frac{[\alpha(1/n)h'(q) - ic'(q)]\{n^2h'(q)g_n(n,i)\alpha'(1/n) - \alpha''(1/n)h(q)\}}{[\alpha(1/n)h'(q) - ic'(q)]^2n^4}\\
&- \frac{[\alpha'(1/n)h(q)]\left\{n^2\alpha(1/n)h''(q)g_n - h'(q)\alpha'(1/n) + 2n[\alpha(1/n)h'(q) - ic'(q)] - n^2ic''(q)g_n\right\}}{[\alpha(1/n)h'(q) - ic'(q)]^2n^4}.
\end{aligned}$$

Since $h'(q^*) = 0$ and noting i^* can be rewritten as

$$i^* = \frac{\alpha'(1/n^*)h(q^*)}{c(q^*)n^*},$$

we have

$$\begin{aligned}
g_n(n^*,i^*) &= -\frac{c(q^*)}{n^*c'(q^*)} < 0;\\
g_i(n^*,i^*) &= -\frac{c(q^*)}{i^*c'(q^*)} < 0;\\
g_{ni}(n^*,i^*) &= \frac{c(q^*)}{i^*n^*c'(q^*)} + \frac{c(q^*)^2[\alpha(1/n^*)h''(q^*) - i^*c''(q^*)]}{(i^*)^2n^*[c'(q^*)]^3};
\end{aligned}$$

$$\begin{aligned}
g_{nn}(n^*,i^*) &= \frac{\alpha''(1/n^*)h(q^*)}{i^*(n^*)^4c'(q^*)}\\
&+ \frac{\alpha'(1/n^*)h(q^*)\left[\frac{n^*c(q^*)}{c'(q^*)}[\alpha(1/n^*)h''(q^*) - i^*c''(q^*)] + 2n^*i^*c'(q^*)\right]}{(i^*)^2[c'(q^*)]^2(n^*)^4}\\
&= \frac{i^*\alpha''(1/n^*)h(q^*)c'(q^*) + \alpha'(1/n^*)h(q^*)\left[\frac{n^*c(q^*)}{c'(q^*)}[\alpha(1/n^*)h''(q^*) - i^*c''(q^*)] + 2n^*i^*c'(q^*)\right]}{(i^*)^2[c'(q^*)]^2(n^*)^4}
\end{aligned}$$

Thus,

$$f(n^*,i^*) = i^*[c[g(n^*,i^*)] + n^*c'[g(n^*,i^*)]g_n(n^*,i^*)] = 0.$$

Moreover for $i^* > 0$, the second partial derivatives are

$$\begin{aligned}
f_n(n^*, i^*) &= i^* \left\{ n^* c''[g(n^*, i^*)] g_n^2(n^*, i^*) + n^* c'[g(n^*, i^*))] g_{nn}(n^*, i^*) + 2c'[g(n^*, i^*)] g_n(n^*, i^*) \right\} \\
&= \frac{i^* c''(q^*) c(q^*)^2}{n^* [c'(q^*)]^2} + \frac{\alpha''(1/n^*) h(q^*) c'(q^*)}{(n^*)^3 c'(q^*)} \\
&+ \frac{\alpha'(1/n^*) h(q^*) \left\{ \frac{n^* c(q^*)}{c'(q^*)} [\alpha(1/n^*) h''(q^*) - i^* c''(q^*)] \right\}}{i^* c'(q^*)(n^*)^3} + 2\frac{i^* c(q^*)}{n^*} - 2\frac{i^* c(q^*)}{n^*} \\
&= \frac{i^* c''(q^*) c(q^*)^2}{n^* [c'(q^*)]^2} + \frac{\alpha''(1/n^*) h(q^*)}{(n^*)^3} + \frac{\alpha'(1/n^*) h(q^*) \left\{ \frac{n^* c(q^*)}{c'(q^*)} [\alpha(1/n^*) h''(q^*) - i^* c''(q^*)] \right\}}{i^* c'(q^*)(n^*)^3} \\
&= \frac{[\alpha'(1/n^*) h(q^*)]^2 c''(q) - [\alpha'(1/n^*) h(q^*)]^2 c''(q)}{i [c'(q)]^2 (n^*)^3} \\
&+ \frac{\alpha''(1/n^*) h(q^*)}{(n^*)^3} + \frac{\alpha'(1/n^*) h(q^*) \left\{ \frac{n^* c(q^*)}{c'(q^*)} [\alpha(1/n^*) h''(q^*)] \right\}}{i^* c'(q^*)(n^*)^3} \\
&= \underbrace{\frac{\alpha''(1/n^*) h(q^*)}{(n^*)^3}}_{(-)} + \underbrace{\frac{\alpha'(1/n^*) h(q^*) \left\{ \frac{n^* c(q^*)}{c'(q^*)} [\alpha(1/n^*) h''(q^*)] \right\}}{i^* c'(q^*)(n^*)^3}}_{(-)} < 0.
\end{aligned}$$

$$\begin{aligned}
f_i(n^*, i^*) &= i^* \left\{ c'[g(n^*, i^*)] g_i(n^*, i^*) + n^* c''[g(n^*, i^*)] g_n(n^*, i^*) g_i(n^*, i^*) + n^* c'[g(n^*, i^*)] g_{ni}(n^*, i^*) \right\} \\
&+ \underbrace{c[g(n^*, i^*)] + n^* c'[g(n^*, i^*)] g_n(n^*, i^*)}_{=0} \\
&= -c(q^*) + \frac{c''(q^*) c(q^*)^2}{c'(q^*)^2} + i^* n^* c'(q^*) g_{ni}(n^*, i^*)) \\
&= -c(q^*) + c(q^*) + \frac{c''(q^*) c(q^*)^2}{c'(q^*)^2} + \frac{c(q^*)^2}{i^* [c'(q^*)]^2} [\alpha(1/n^*) h''(q^*) - i^* c''(q^*)] \\
&= \frac{c''(q^*) c(q^*)^2}{c'(q^*)^2} + \frac{c(q^*)^2}{i^* [c'(q^*)]^2} [\alpha(1/n^*) h''(q^*) - i^* c''(q^*)] \\
&= \frac{i^* c''(q^*) c(q^*)^2 - i^* c''(q^*) c(q^*)^2}{i^* [c'(q^*)]^2} + \frac{c(q^*)^2 \alpha(1/n^*) h''(q^*)}{i^* [c'(q^*)]^2} \\
&= \underbrace{\frac{c(q^*)^2 \alpha(1/n^*) h''(q^*)}{i^* [c'(q^*)]^2}}_{(-)} < 0.
\end{aligned}$$

Since $f(n^*, i^*) = 0$ and $f_n(n^*, i^*) < 0$, by the Implicit Function Theorem, there exists an open neighborhood $(n_0, n_1) \times (i_0, i_1)$ around the first-best (n^*, i^*) and a continuously differentiable

function, $n^p : (i_0, i_1) \to (n_0, n_1)$ such that for $i \in (i^*, i_1)$, the function $n^p(i)$ gives the unique value of $n \in (n_0, n_1)$ such that

$$f[n^p(i), i] = 0.$$

Because $f_n(n^*, i^*) = 0$ and because f is continuously differentiable, the objective function is locally concave and hence for some $i_2 \in (i^*, i_1]$, $n^p(i)$ is the local maximizer for the objective function. Following similar arguments as those used in Proposition 2 (2), we can also show that $n^p(i)$ achieves maximum globally. Moreover, by the IFT again, for all $i \in (i^*, i_2]$,

$$\frac{d}{di} n^p(i) = -\frac{f_i[n^p(i), i]}{f_n[n^p(i), i]}.$$

Because $f_n(n^*, i^*) < 0$ and $f_i(n^*, i^*) < 0$, and because f is continuously differentiable, there exists $\bar{i}' \in (i^*, i_2]$ for which if $i \in (i^*, \bar{i}']$, then $f_n[n^p(i), i] < 0$ and $f_i[n^p(i), i] < 0$ and hence $\frac{d}{di} n^p(i) < 0$.

Finally, $(q^p(i), n^p(i)) = (g(n^p(i), i), n^p(i))$ is a constrained-efficient outcome follows directly from Claim 0 and Claim 1. □

Claim 2 shows that if we impose the constraint that $k = 0$, then for a range of inflation rates above i^*, there is a unique constrained-efficient outcome for each i in that range with the number of buyers entering the DM decreases with i. The next two claims show that the constraint $k = 0$ is binding.

Claim 3. Consider the economy with capital and consider the maximization problem (40) subject to (41) and (42). Suppose that $i > i^*$. Then, the seller's participation constraint, (42), binds at the optimum and $q^p < q^*$ at the optimum.

Proof. Consider the Lagrangian associated with (40) subject to (41), (42), $z \geq 0$, and $k \geq 0$:

$$\begin{aligned}
\mathcal{L}(q, z, k, n; \lambda, \mu, \nu_z, \nu_k) &= n\alpha(1/n)[u(q) - c(q)] - nv - n(1 + r - A)k \\
&+ \lambda\{-iz - (1 + r - A)k + \alpha(1/n)[u(q) - z - Ak] - v\} \\
&+ \mu\{-c(q) + z + Ak\} \\
&+ \nu_z z + \nu_k k,
\end{aligned}$$

where λ, $\mu \geq 0$, $\nu_z \geq 0$, and $\nu_k \geq 0$ are the Lagrange multipliers associated with (41), (42), $z \geq 0$, and $k \geq 0$ respectively. The first-order necessary conditions with respect to q, z, k, n and the

complementary slackness conditions for $z \geq 0$ and $k \geq 0$ are respectively

$$[\alpha(1/n^p)(n^p + \lambda)]u'(q^p) - [n^p\alpha(1/n^p) + \mu]c'(q^p) = 0 \tag{104}$$

$$\lambda[i + \alpha(1/n^p)] = \mu + \nu_z \tag{105}$$

$$-(1 + r - A)(n^p + \lambda) + A(\mu - \lambda\alpha(1/n^p)) + \nu_k = 0 \tag{106}$$

$$\left[\alpha(1/n^p) - \alpha'(1/n^p)/n^p\right][u(q^p) - c(q^p)] - (1 + r - A)k^p$$
$$-\lambda[\alpha'(1/n^p)/(n^p)^2][u(q^p) - z^p - Ak^p] = v \tag{107}$$

$$\nu_z z^p = 0 \tag{108}$$

$$\nu_k k^p = 0. \tag{109}$$

(i) To show that (42) binds for all $i > i^*$, i.e. $\mu > 0$, we consider two cases.
(a) At the optimum, $k^p > 0$. Suppose by contradiction that (42) does not bind, i.e. $\mu = 0$. Consider (z', k') such that $0 \leq k' < k^p$ and $z' + Ak' = z^p + Ak^p$. By continuity, the allocation (q^p, z', k', n^p) still satisfies the seller's participation constraint, (42). But since $k' < k^p$, this leads to higher welfare, a contradiction.
(b) At the optimum, $k^p = 0$. Suppose by contradiction that (42) does not bind, i.e. $\mu = 0$. Then (42) holds with strict inequality and given $k^p = 0$, we have $z^p > 0$. Hence $\nu_z = 0$ and from (105), $\lambda = 0$. Hence $q^p = q^*$, $n^p = n^*$, and $k^p = 0$, which implies that the first-best is implementable for $i > i^*$, a contradiction. Hence (42) binds.

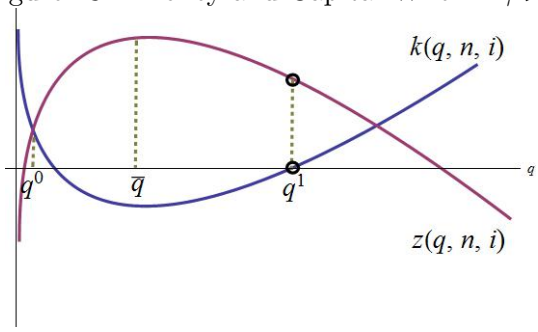
We now show that that $q^p < q^*$ for all $i > i^*$. First note that $q^p \neq q^*$ unless $\lambda = 0$ and $\nu_z = 0$ which is violated when $i > i^*$. Now suppose $q^p > q^*$ and consider a deviation such that q^p decreases to q^* and real balances are reduced to $z' = z^p - [u(q^p) - u(q^*)] \geq c(q^*)$. As $z' < z^p$, this deviation leads to higher welfare and is incentive feasible, a contradiction. Hence $q^p < q^*$ for $i > i^*$.

□

With (41) and (42) at equality, the problem becomes

$$\max_{(q,z,k,n)} n\alpha(1/n)[u(q) - c(q)] - nv - n(1 + r - A)k$$
subject to
$$-iz - (1 + r - A)k + \alpha(1/n)[u(q) - c(q)] = v$$
$$z + Ak = c(q).$$

Figure 43: Money and Capital When $A\gamma > 1$



Claim 4. For any $i \in (i^*, i^0]$, where $i^0 = \frac{\alpha(1/n^*)[u(\overline{q})-c(\overline{q})]-v}{c(\overline{q})} > i^*$, $\overline{q} \leq q^*$ solves

$$\frac{u'(\overline{q})}{c'(\overline{q})} = 1 + \left(\frac{1+r-A}{A}\right),$$

and for any constrained-efficient outcome $(q^p(i), z^p(i), k^p(i), n^p(i))$, $k^p(i) = 0$.

Proof. (a) First consider the case where $A\gamma \leq 1$, or, equivalently, $A(i+1) \leq 1+r$. This means the rate of return on capital, $F'(k) = A$, is less than or equal to the rate of return on fiat money, γ^{-1}. Suppose that $k^p(i) > 0$. Then, let $k' = k^p(i) - \epsilon > 0$ and let $z' = z^p(i) + [(1+r-A)\epsilon]/i$. Then $(q^p(i), z', k', n^p(i))$ satisfies (41) and (42). Notice that $-iz^p(i)-(1+r-A)k^p(i) = -iz'-(1+r-A)k'$ by construction and

$$z' + Ak' = z^p(i) + Ak^p(i) + [1+r-(1+i)A]\epsilon/i \geq z^p(i) + Ak^p(i)$$

because $1+r-(1+i)A \geq 0$. Obviously the new outcome, $(q^p(i), z', k', n^p(i))$, is welfare-improving. (b) Suppose now that $A\gamma > 1$ so that capital has a higher rate of return than fiat money. Given i and v and a choice of q and n, (41) and (42) at equality implies a unique solution for z and k given

by

$$
\begin{aligned}
z(q,n,i) &= \frac{-(1+r-A)c(q)+A\alpha(1/n)[u(q)-c(q)]-Av}{-(1+r)+A(1+i)} \\
&= \beta\left\{\frac{-(1+r-A)c(q)+A\alpha(1/n)[u(q)-c(q)]-Av}{A\gamma-1}\right\}, \\
k(q,n,i) &= \frac{-\alpha(1/n)[u(q)-c(q)]+ic(q)+v}{-(1+r)+A(1+i)} \\
&= \beta\left\{\frac{-\alpha(1/n)[u(q)-c(q)]+ic(q)+v}{A\gamma-1}\right\}.
\end{aligned}
$$

With $A\gamma > 1$ and $v > 0$, $z(q,n,i)$ is a concave function of q while $k(q,n,i)$ is a convex function of q as illustrated in Figure 43. Given i, using the fact that (41) holds at equality, the objective function simplifies to a choice of q and n:

$$\max_{q,n} n \cdot i \cdot z(q,n,i)$$

subject to $k(q,n,i) \geq 0$. Consider first the q^0 such that $k(q^0,n,i) = z(q^0,n,i)$. Then, $z(q,n,i) = c(q^0)/(A+1)$. While for q^1 such that $k(q^1,n,i) = 0$, $z(q,n,i) = c(q^1) > c(q^0)/(A+1)$ as $q^0 < \bar{q}$. Now we show that for all $n \in [0,n^*]$, and hence $\alpha(1/n) \in [\alpha(1/n^*),1]$, $z_q(q,n,i) > 0$ if $q > q^1$. Now,

$$z_q(q,n,i) = \beta\left\{\frac{-(1+r-A)c'(q)+A\alpha(1/n)[u'(q)-c'(q)]}{A\gamma-1}\right\},$$

and hence it suffices to show that $z_q(q^1,n,i) \leq 0$, that is,

$$-(1+r-A)c'(q^1)+A\alpha(1/n)[u'(q^1)-c'(q^1)] \leq 0.$$

It suffices to show that $q^1 \geq \bar{q}$, that is, $k(\bar{q},n,i) \geq 0$, as for all $n \in [0,n^*]$,

$$-(1+r-A)c'(\bar{q})+A\alpha(1/n)[u'(\bar{q})-c'(\bar{q})] \leq -(1+r-A)c'(\bar{q})+A[u'(\bar{q})-c'(\bar{q})] = 0.$$

Suppose by contradiction that $\overline{q} > q^1$. Then

$$-\alpha(1/n)[u(\bar{q})-c(\bar{q})]+ic(\bar{q})+v \leq -\alpha(1/n^*)[u(\bar{q})-c(\bar{q})]+ic(\bar{q}) \leq 0$$

as $i \leq i^0$. Notice that $i^0 > i^*$ because, as we have assumed,

$$v < \frac{\alpha(1/n^*)[c(q^*)(u(\bar{q})-c(\bar{q}))-c(\bar{q})(u(q^*)-c(q^*))]}{c(q^*)-c(\bar{q})}.$$

□

By Claim 2 and Claim 4, and if we take $\bar{i} = \min\{i^0, \bar{i}'\}$, then for all $i \in (i^*, \bar{i}]$, there is a unique constrained-efficient outcome and $k^p(i) = 0$, $\frac{d}{di} n^p(i) < 0$.

(3) Because $A = 0$, by Claim 0, a constrained-efficient outcome satisfies

$$\alpha(1/n)[u(q) - c(q)] \geq ic(q) + v.$$

For each $i \in \mathbb{R}_+$, let $\tilde{q}_i$ solves

$$u'(\tilde{q}_i) - c'(\tilde{q}_i) = ic'(\tilde{q}_i).$$

By concavity of u and convexity of c, one can verify that $\tilde{q}_i$ decreases with i and $\tilde{q}_i \to 0$ as $i \to \infty$. Then, for all $q \in \mathbb{R}_+$ and for all $n \in [0, 1]$,

$$\alpha(1/n)[u(q) - c(q)] - ic(q) \leq [u(\tilde{q}_i) - c(\tilde{q}_i)] - ic(\tilde{q}_i).$$

Let $\bar{i}$ be such that

$$[u(\tilde{q}_{\bar{i}}) - c(\tilde{q}_{\bar{i}})] = v.$$

Then, if $i > \bar{i}$, for all $n \in [0, 1]$ and for all $q \in \mathbb{R}_+$,

$$\alpha(1/n)[u(q) - c(q)] - ic(q) \leq [u(\tilde{q}_{\bar{i}}) - c(\tilde{q}_{\bar{i}})] - \bar{i}c(\tilde{q}_{\bar{i}}) < v.$$

So the only feasible allocation is autarky.

(4) By (3), if $i > \bar{i}$ and if $k^p = 0$, then the outcome must be autarky. However, because $\mathcal{W}^c > 0$, there is an outcome with $k^p > 0$ with welfare $\mathcal{W}^c$. So we must have $k^p > 0$.

Walrasian Price-Taking

In our baseline model, we assume that buyers and sellers meet pairwise in the decentralized market and look for individually rational and coalition-proof mechanisms that maximize social welfare. Here we consider a notion of ex-post competition similar to Lucas and Prescott (1974) and Rocheteau and Wright (2005) where individuals trade in large groups against market-clearing prices. Due to the equivalence between the core and competitive equilibrium, the allocation achieved through competitive markets is the only coalition-proof allocation. In what follows, we show another example of a mechanism that is consistent with the core requirement and also features non-monotone search intensity.

While the presence of competitive price-taking makes the first stage a centralized meeting, we will still refer to it as the DM. To maintain a role for money in the DM, we continue to assume a double-coincidence problem, private information over individual trading histories, lack of commitment, and no record-keeping. To capture search frictions with Walrasian pricing, we assume that there are competitive marketplaces (or trading posts) present in the DM but it takes time for individuals to be able to trade. For instance, buyers can shop around but it takes effort $e \in [0,1]$ that costs $\psi(e) \in [0,\infty)$ in terms of disutility to shop. Letting $\theta \equiv 1/\overline{e}$ as before, the probability of trading in the marketplace is therefore $e\alpha(\theta)$ for a buyer and $\alpha(\theta)/\theta$ for a seller. Hence in the DM, the average time it takes for a buyer to trade ("shopping time") is $1/[e\alpha(1/\overline{e})]$.[25]

The trading procedure in the DM will operate as follows. We denote by p the price of the DM good in terms of the CM numéraire. At the beginning of the DM, the market price p is posted and observable to all agents. Given the price p, buyers indicate how much they want to buy, q^b, while sellers indicate how much they want to sell, q^s.

Competitive Equilibrium

An active seller in the DM chooses a quantity to supply, q^s, that maximizes their DM surplus:

$$q^s = \arg\max_q \{-c(q) + pq\}.$$

Hence, the relative DM price must equal the seller's marginal cost of production:

$$p = c'(q^s).$$

Upon entering the DM, the buyer chooses search effort e by incurring a cost $\psi(e)$. In turn, the buyer is active in the first stage with probability $e\alpha(\theta)$ and demands q^b units of goods. Taking as given the market price p, the buyer's value function in the DM is

$$V^b(z) = \max_{e\in[0,1]} \left\{ -\psi(e) + e\alpha(\theta) \max_{q^b\in[0,q^*]} [u(q^b) - pq^b] + W^b(z) \right\},$$

[25] As in our baseline model, the dependence of matching probabilities on market tightness θ reflects the presence of search externalities. Alternatively, a specification without search externalities (e.g. the probability of trading for a buyer is e while the probability of trading for seller is fixed at one) would be similar to shopping-time models such as McCallum and Goodfriend (1987) and Lucas (2000). Here we consider the model with search frictions to make the environment more easily comparable with the baseline model. See also Rocheteau and Wright (2005) for a similar formalization of Walrasian trade in the DM where buyers and sellers must wait in a line to enter a competitive marketplace.

subject to

$$pq^b \leq z.$$

The buyer chooses the quantity of goods to purchase in the DM, or equivalently, the amount of real balances to bring into the DM. The buyer's problem is

$$(e, q^b) = \arg \max_{e \in [0,1], q \in [0,q^*]} \{-iz - \psi(e) + e\alpha(\theta)[u(q) - pq]\} \tag{110}$$

subject to $pq^b \leq z$.

The market-clearing condition for the first stage implies that market price p must be such that the aggregate demand for the DM good equals the aggregate supply:

$$\int_{b \in [0,1]} e_b \alpha(1/\overline{e}) q^b \, db = \overline{e}\alpha(1/\overline{e}) q^s,$$

where e_b is the search intensity of an individual buyer $b \in [0,1]$. In a stationary symmetric equilibrium where real balances are constant over time and all buyers choose the same search intensity and real balances, market-clearing implies $q^b = q^s \equiv q$. Provided that $\gamma > \beta$, the buyer's cash constraint binds and hence buyers spend all their cash, $z = pq$. Since $p = c'(q)$, real balances are given by $z = c'(q)q$. So long as $\gamma > \beta$, sellers do not have a strict incentive to take real balances into the DM.

Lemma 5. *Given (p, i, θ), the buyer's optimal choice of (e, q) is characterized by the correspondence $D(p, i, \theta)$ defined by*

$$D(p, i, \theta) = \arg \max_{e,q} \{-ipq - \psi(e) + e\alpha(\theta)[u(q) - pq]\}.$$

The correspondence $D(p, i, \theta)$ is non-empty, compact-valued, and upper-hemi continuous. Moreover, $D(p, i, \theta)$ is decreasing in (p, i).

Proof. Define $\Omega(e, q; p, i, \theta) \equiv \{-ipq - \psi(e) + e\alpha(\theta)[u(q) - pq]\}$ where

$$D(p, i, \theta) = \arg \max_{e,q} \Omega(e, q; p, i, \theta).$$

First note that the buyer's choice of (e, q) lies in a compact set, $[0,1] \times [0, q^*]$. By the Theorem of the Maximum, the correspondence $D(p, i, \theta) : [0, \infty) \times [0, \infty) \times [1, \infty) \to [0,1] \times [0, q^*]$ is non-empty, compact-valued, and upper-hemi continuous.

We show that $D(p, i, \theta)$ is decreasing in (p, i) by applying Topkis' Theorem (see Theorem 2.8.1 in Topkis (1998)). First note that $\Omega(e, q; p, i, \theta)$ is a supermodular function of (e, q), since $p = c'(q)$

implies

$$\Omega_{eq} = \alpha(\theta)[u'(q) - c'(q)] \geq 0.$$

In addition, $\Omega(e, q; p, i, \theta)$ is strictly supermodular in (e, q) if $e < 1$ and $q < q^*$. Next, we show that $\Omega(e, q; p, i, \theta)$ exhibits increasing differences in $(e, q), (-i, -p)$. Note that

$$\Omega_e = -\psi'(e) + \alpha(\theta)[u(q) - pq]$$

and

$$\Omega_q = -ip + e\alpha(\theta)[u'(q) - p]$$

implies the second derivatives

$$\Omega_{ei} = 0, \ \Omega_{ep} = -\alpha(\theta)q \leq 0,$$

$$\Omega_{qi} = -p \leq 0, \ \Omega_{qp} = -[i + e\alpha(\theta)] \leq 0.$$

Since $\Omega(e, q; p, i, \theta)$ is a supermodular function of (e, q) and $\Omega(e, q; p, i, \theta)$ exhibits increasing differences in $(e, q), (-i, -p)$, then by Topkis' Theorem, the optimal choice correspondence $\arg\max_{e,q} \Omega(e, q)$ will be increasing in $(-i, -p)$. Hence $D(p, i, \theta) = \arg\max_{e,q} \Omega(e, q)$ is decreasing in (i, p), and strictly decreasing if $e < 1$ and $q < q^*$. □

From Lemma 5, the buyer's optimal choice set (e, q) are decreasing functions of (i, p) and strictly decreasing if the solution is interior. In what follows, we focus on situations where the buyer's problem admits a unique solution. This will be true at $i = 0$; for i close to zero, the buyer's objective is strictly jointly concave and hence the solution is unique near the Friedman rule. More generally however, the buyer's objective is not concave.

Assuming an interior solution, the buyer's choice of search intensity, e, and demand for the DM good, q^b, solves

$$-\psi'(e) + \alpha(1/e)[u(q) - pq] = 0,$$

$$u'(q) = p\left[1 + \frac{i}{e\alpha(1/e)}\right].$$

Definition. *With Walrasian price-taking, a stationary and symmetric competitive equilibrium is a pair (q, e) such that*

1. *Given (i, p), $(e, q) \in D(i, p, \theta)$,*
2. *The price p solves $p = c'(q)$,*
3. *The quantity traded q clears the market: $\int_{b\in[0,1]} e_b\alpha(1/\overline{e})q^b \, db = \overline{e}\alpha(1/\overline{e})q^s$.*

When the solution is interior, we have

$$\frac{u'(q)}{c'(q)} = 1 + \frac{i}{e\alpha(1/e)}, \tag{111}$$

$$\psi'(e) = \alpha(1/e)\left[u(q) - qc'(q)\right]. \tag{112}$$

According to (111), output traded in the DM approaches q^* as i approaches zero. According to (112), the buyer's choice of search intensity is such that the marginal cost from searching equals the marginal gain in the buyer's surplus.

Proposition 5. *Consider the model with money alone and Walrasian price-taking in the first stage.*

1. *At $i = 0$, the unique solution satisfies $q = q^*$ and e such that*

$$\begin{matrix} e > e^* \\ e = e^* \\ e < e^* \end{matrix} \text{ if } \left[1 - \frac{\alpha'(1/e^*)}{\alpha(1/e^*)e^*}\right] \begin{matrix} < \\ = \\ > \end{matrix} \left[\frac{u(q^*) - q^*c'(q^*)}{u(q^*) - c(q^*)}\right]. \tag{113}$$

2. *Suppose i is close to zero: (*i*) if c is strictly convex ($c'' > 0$), $\frac{dq}{di} < 0$ and $\frac{de}{di} > 0$, (*ii*) if c is linear ($c'' = 0$), $\frac{dq}{di} < 0$ and $\frac{de}{di} < 0$.*

Proof. **(1)** Comparing (18) and (111), the two conditions coincide if and only if $i = 0$, which is the Friedman rule. Comparing (20) and (112), the two conditions coincide if

$$\left[1 - \frac{\alpha'(1/e^*)}{\alpha(1/e^*)e^*}\right][u(q^*) - c(q^*)] = u(q^*) - q^*c'(q^*).$$

Hence at $i = 0$, equilibrium search intensity will typically be inefficient: the Friedman rule generates efficiency in the quantity traded per match but not in the number of trades.
(2) Suppose i is close to zero.
(i) When c is strictly convex ($c'' > 0$), q and e are given by (111) and (112). We apply the Implicit Function Theorem (IFT) to (111) and differentiate to obtain

$$\frac{de}{dq} = \frac{[e\alpha(1/e)]^2[c''(q)u'(q) - u''(q)c'(q)]}{i[\alpha(1/e) - \alpha'(1/e)/e][c'(q)]^2} > 0, \tag{114}$$

since α is strictly increasing, u is strictly increasing and concave for $q > 0$, and c is strictly increasing

and convex. Similarly from (112),

$$\frac{de}{dq} = \frac{\alpha(1/e)^2 \left[u'(q) - c'(q) - qc''(q)\right]}{\left[\alpha(1/e)\psi''(e) + \alpha'(1/e)\psi'(e)/e^2\right]}. \tag{115}$$

When i is close to zero, q is in the neighborhood of q^*, and from (115), $-q^*c''(q^*) \leq 0$ and hence $\frac{de}{dq} \leq 0$ at $q = q^*$. Hence for q close to q^*, e given by (112) is decreasing in q. Consequently, an increase in i leads to a fall in q from Lemma 5 and a rise in e from (115).
(ii) When c is linear, $\frac{de}{dq} = 0$ from (115) for all $q \in [0, q^*]$. Hence from Lemma 5, an increase in i unambiguously leads leads to a fall in q and e for all $i \geq 0$. □

While price-taking generates the efficient level of output under the Friedman rule, it fails to internalize the congestion externality in the DM stage. Equilibrium under price-taking is therefore generically inefficient.

When the seller's cost of production is strictly convex, and hence the marginal cost of production is strictly increasing, the price of the DM good in units of numeraire, $p = c'(q)$, falls with inflation. Consequently, $z = c'(q)q$ implies the buyer's surplus, $u(q) - qc'(q)$, is increasing in q when q is small but decreasing when q is large. Close to the Friedman rule, an increase in inflation induces a rise in the buyer's surplus, and from (112), induces a rise the buyer's search effort. Since an increase in inflation (close to the Friedman rule) can increase search intensity under a convex cost function, the frequency of trades, $e\alpha(1/e)$, also rises. The effect of inflation on aggregate output, $\mathcal{Q} \equiv e\alpha(1/e)q$ however is generally ambiguous since DM output per trade, q, falls with inflation. If the extensive margin effect of the increase in trading frequency outweighs the intensive margin effect of the fall in output per trade, then low inflation can produce an overall rise in DM aggregate output.

If instead the seller's production cost is linear, then the buyer's surplus, $u(q) - q$, is strictly increasing in q for $q < q^*$. Since the marginal cost of production is constant, the price of the search good, p, is independent of the cost of holding money, i. Without this price effect, inflation induces a fall in both DM output, q, and search intensity, e. Notice also that no matter the assumption on the seller's production cost, for i large enough, both output and search intensity eventually fall.

Interestingly, since Walrasian pricing does not internalize the search externality even at the Friedman rule, an increase in inflation can be welfare improving when search intensity is inefficiently low. This requires a convex production cost which generates a marginal cost pricing rule. However when the seller's cost of production is linear, then inflation will unambiguously decrease welfare.