The Phillips curve in a matching model

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Abstract

Shocks to the stock of money that are observed with a one-period lag are introduced into the random matching model of money of Shi (1995) and Trejos-Wright (1995), a model designed to show that money helps facilitate trade. The shock follows a finite-state Markov process. It is shown that if the shock is *almost* perfectly persistent, then the equilibrium gives rise to a Phillips curve, a positive association between the current stock of money and the level of per capita output. Moreover, that association is stronger than in the same model without an information lag.

Key words: Phillips curve, incomplete information, monetary search.

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1 Introduction

In his Nobel lecture [7] entitled "Monetary Neutrality," Lucas begins by describing Hume's [2] views about the effects of changes in the money supply. Lucas emphasizes that Hume's views were dependent on how changes in the

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quantity of money come about. In order to get neutrality, Hume set out very special conceptual experiments which, when correct, amount to changes in monetary units. Hume's view about the effects of other kinds of changes in the quantity of money, are set out in the following passage.

> Accordingly we find that, in every kingdom into which money begins to flow in greater abundance than formerly, everything takes a new face: labour and industry gain life, the merchant bcomes more enterprising...
>
> To account, then, for this phenomenon, we must consider, that though the high price of commodites be a necessary consequence of the encrease of gold and silver, yet it follows not immediately upon that encrease, but some time is required before the money circulates through the whole state, and makes its effect be felt on all ranks of people. At first, no alteration is perceived; by degrees the price rises, first of one commodity, then of another, till the whole at last reaches a just proportion with the new quantity of specie in the kingdom. In my opinion, it is only in this interval or intermediate situation, between the acquisition and rise of prices, that the encreasing quantity of gold and silver is favorable to industry. When any quantity of money is imported into a nation, it is not at first dispersed into many hands but is confined to the coffers of a few persons, who immediately seek to employ it to advantage. [Hume [2], page 37.]

This passage asserts that there is a Phillips curve, a positive association between increases in the stock of money and the level of real economic activity, and offers what may at some time have been regarded as an explanation of it. A modern economist would not regard Hume's discussion as an explanation, but might look to it for hints about modeling ingredients that when rigorously analyzed could conceivably constitute an explanation.

The passage contains at least two hints about modeling ingredients. First, changes in the quantity of money come about in a way that gives rise to changes in relative money holdings among people. In particular, the changes for individuals are not proportional to initial holdings, as is required for neutrality. Second, trade seems to be occurring within small groups rather than in a centralized market. That suggests the use of some sort of search/matching model. Given those ingredients, the passage hints at two conjectures that might be studied. One is that a change in relative money holdings has real effects that dissipate over time through the effects of subsequent trades on those relative holdings. The other is that the change occurs in a way that is not seen by everyone when it occurs and that the real effects dissipate when people learn about it. Although these are not mutually exclusive conjectures, we pursue only the second here. In order to study it, we assume that the aggregate shocks to relative money holdings are observed with a one-period lag and we contrast that version of our model with one in which there is no information lag.

This is not the first attempt to use a search/matching model to study the effects of monetary shocks. Indeed, our formulation is closely related to Wallace [13] and to Katzman *et al.* [4]. Wallace [13] studies a one-time permanent shock, which is simple because the distribution of future continuation values is exogenous and because the presence of the shock does not have incentive effects. Here, as in Katzman *et al.* [4], there are ongoing shocks. Katzman *et al.* [4] assume that some fraction of the population sees aggregate shocks as they occur and that gains and losses of money are concentrated among them. We work with a version that is conceptually simpler in two respects. First, everyone is subject to shocks to their money holdings. Second, while preserving asymmetric information *across* meetings, there is no asymmetric information *within* meetings. One benefit of our formulation is that we are able to get a general qualitative result for the Phillips curve when the monetary shocks are rare. That is, we get a Phillips curve—both absolutely and relative to the no-information lag version—when the probability of a change in the stock of money is sufficiently small.

The background setting is the random-matching model of Shi [9] and Trejos-Wright [10] in which individual money holdings are limited to the set $\{0, 1\}$. The aggregate shock follows an exogenous finite-state Markov process and is accomplished by way of random losses or gains of money by individuals.[1] With $\{0, 1\}$ money holdings, trade occurs only in *trade-meetings*, meetings in which the potential consumer (buyer) has money and the potential producer (seller) does not.

Our results depend on two parameter restrictions that are easily described in the context of the model without shocks. First, we assume that the preference and meeting-rate parameters are such that the constrained efficient outcome would improve if interest could be paid on money or, more simply, if barren money were replaced by a dividend-bearing asset (a Lucas tree)—an improvement that goes beyond consumption of the dividend and is in keeping with standard views about the desirability of paying interest on money. Second, we assume that the shocks occur around the optimal stock of money in the absence of shocks. Under the first assumption, the optimal stock has less than half of the population with money. It implies that the fraction of meetings that are *trade meetings* is increasing in the stock at the optimal stock. In other words, we choose a support for the random stock of money so that an increase in the stock of money has a positive extensive-margin effect—whether or not there is an information lag. However, only in the model with the information lag can we be sure that the positive extensive margin effect is not fully offset by an intensive-margin effect. We also assume buyer-take-it-or leave-it offers, the bargaining protocol that supports the constrained efficient trade when there are no shocks.

[1]There are several ways to interpret the shocks. If money consists of sea-shells, then some people with a shell lose it and some people without one find one. In a fiat-money world in which a government controls the stock of money, the shocks should be viewed as shocks to taxes and transfers in the form of money.

Although we have motivated our choice of model by way of hints we find in Hume's discussion, our main message is that we get a Phillips curve with few assumptions beyond those required to give money a role. The model we use was devised solely to demonstrate such a role. One ingredient of it is a depiction of a double-coincidence problem. Some version of such a problem is necessary for a role for money and such problems have almost always been exposited by reference to pairwise meetings. The second main ingredient is dispersed information across the meetings in the model. That implies privacy of individual trading histories, some degree of which is also necessary for a role for money (see Ostroy [8], Townsend [11], and Kocherlakota [5]). Crucially, that same dispersed information prevents people from seeing and aggregating individual shock-realizations across meetings in a way that would allow them to see the aggregate shock when it occurs. As we demonstrate below, besides the aforementioned parameter restrictions, that is all we need to get a Phillips curve.

2 The model

Time is discrete and there is a nonatomic measure of people, each of whom maximizes expected discounted utility with discount factor $\beta \in (0, 1)$. Production and consumption occur in pairwise meetings that occur at random in the following way. Just prior to such meetings, each a person looks forward to being a consumer (a buyer) who meets a random producer (seller) with probability σ, looks forward to being a producer who meets a random consumer with probability σ, and looks forward to no pairwise meeting with probability $1 - 2\sigma$, where $\sigma \leq 1/2$.[2] The period utility of someone who becomes a consumer and consumes $y \in \mathbb{R}_+$ is $u(y)$, where u is strictly increasing, concave, twice differentiable, and satisfies $u(0) = 0$. The period utility of someone

[2] If $\sigma = 1/K$ for an integer $K > 2$, then, as is well-known, it can be interpreted as the number of goods and specialization types in Shi [9] and in Trejos-Wright [10].

who becomes a producer and produces $y \in \mathbb{R}_+$ is $-c(y)$, where c is strictly increasing, convex, differentiable, and satisfies $c(0) = 0$, and there exists $\bar{y}$ such that $\lim_{y \to \bar{y}} c(y) = \infty$. In addition, $u(y) - c(y)$ is strictly concave in y and either $u'(0) = \infty$ or $c'(0) = 0$. Production is perishable; it is either consumed or lost. Throughout, we assume that individual money holdings are in the set $\{0, 1\}$ and that the only possibility for trade involves an exchange of some amount of the good for (a probability of receiving) money.

The above is the model in Shi [9] and in Trejos-Wright [10]. We amend it by introducing aggregate and individual shocks to money holdings. The state entering a date is the fraction of agents with a unit of money and the state is common knowledge. It is denoted $m \in M$, where $M = \{m_1, m_2, ..., m_N\}$ with $m_i < m_{i+1}$ and $m_1 > 0$. Then, there is an aggregate shock that determines next period's state, $m' \in M$. This shock is determined by an exogenous Markov process on the set M with transition function τ, an $N \times N$ matrix, where $\tau_{mm'}$ is the probability that tomorrow's state is m' given that today's state is m. We consider two versions that differ in terms of agents' information about the aggregate shock. In the version with an information lag, m' is revealed at the end of the period; in the no information-lag version, it is observed when it occurs. Then, in both versions, people meet in pairs at random and, at the same time, individual gains (transfers) of money and losses of money to people in meetings are realized in a way that accomplishes the aggregate shock. Both people in a meeting see their own and their trading partner's realizations, which are determined independently of each other.[3] Then trade occurs via consumer take-it-or-leave-it offers.

For any inherited state m and any current state m', let $p_+(m, m')$ be the probability that an agent without money is given money and let $p_-(m, m')$

[3]The above timing produces symmetric information in each meeting and precludes risk-sharing within a meeting as would be possible if meetings occur before individuals gains and losses of money are realized. We suspect that a version with risk-sharing within meetings would have similar implications for aggregates.

be the probability that an agent with money loses it. We assume that

$$[p_-(m,m'),p_+(m,m')] = \begin{cases} (1-\frac{m'}{m}+\varepsilon, \frac{m}{1-m}\varepsilon) \text{ if } m' \le m, \\ \\ (\varepsilon, \frac{m'-m}{1-m}+\frac{m}{1-m}\varepsilon) \text{ if } m' \ge m, \end{cases} \tag{1}$$

where $\varepsilon \in (0, 1-m_N)$. It follows that these individual shocks accomplish the transition from m to m'; that is,

$$m'-m = \begin{cases} (1-m)\frac{m}{1-m}\varepsilon - m[1-\frac{m'}{m}+\varepsilon] \text{ if } m' \le m, \\ \\ (1-m)[\frac{m'-m}{1-m}+\frac{m}{1-m}\varepsilon] - m\varepsilon \text{ if } m' \ge m, \end{cases}$$

where on each line, the first term is the measure of those who gain money, $(1-m)p_+(m,m')$, and the second term is the measure of those who lose money, $mp_-(m,m')$. This formulation of individual shocks is such that the support of post-transfer money holdings for any person is $\{0,1\}$ for all (m,m'). Thus, it conforms to the usual way of specifying individual uncertainty so that an individual realization does not determine an aggregate state.[4]

Throughout, we make the following two assumptions.

A1. The functions u, c, and the scalars β and σ are such that

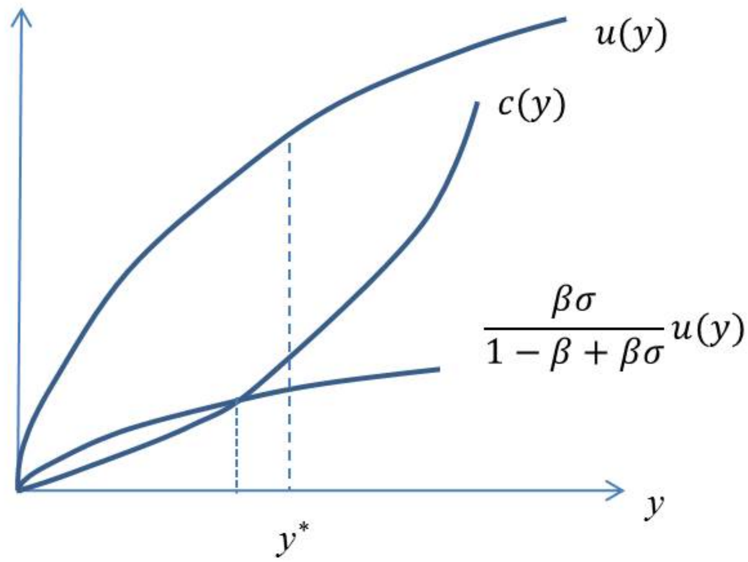
$$c(y^*) > \frac{\beta\sigma}{1-\beta+\beta\sigma}u(y^*), \tag{2}$$

where $y^* = \arg\max_{y\ge 0}[u(y)-c(y)]$. (See Figure 1 for the depiction of $u(y)$ and $c(y)$ that satisfy this and our other assumptions.)

This assures that any output as large as y^*, the first best, does not satisfy the producer's participation constraint even in a version in which past actions

[4] If $\varepsilon = 0$ and no other parameters are introduced, then (1) is the unique scheme consistent with the aggregate shocks. However, if $\varepsilon = 0$, then anyone who receives money knows that $m' > m$ and vice versa for anyone who loses money—inferences we want to rule out.

Figure 1: Assumption A1



are common knowledge or even when the per capita stock of money is near zero. As is well-known, it is sufficient to ensure that better outcomes could be achieved if interest were paid on money.

A2. The support of the aggregate shock, M, satisfies $M \subset (0, 1/2)$.

This assures that the fraction of meetings that are trade meetings, $m(1-m)$, is increasing in the realization of the stock of money, a positive extensive-margin effect.

3 Stationary equilibrium

The state of the economy is the known inherited stock of money, denoted m, whether or not there is an information lag. However, the *state of a meeting* is different in the two versions. We let z^i for $i \in \{0, 1\}$ denote the state of a meeting when the information lag is i. If $i = 0$, the no information lag version, then $z^0 \in Z^0 = M$, the current realized aggregate state. If $i = 1$, then $z^1 = (m, b, s) \in Z^1 = M \times \{n, o\} \times \{n, o\}$, where m is the known inherited state and $b, s \in \{new(n), old(o)\}$ describes the status of the buyer and the seller, respectively. Here, *new* means that the person realized a gain or loss of money in the current meeting and *old* means that the person did not. Given a state z^i, $\gamma^i(\cdot; z^i, \tau)$ denotes the shared posterior on M of the two people in a meeting, a posterior over the current aggregate state. The posterior $\gamma^0(\cdot; z^0, \tau)$ is degenerate on the current realized state; the posterior $\gamma^1(\cdot; z^1, \tau)$ is given by Bayes rule and depends only on the known beginning-of-period state, m, on τ, and on the realized individual transfers implied by (1). (See the appendix for the exact expressions for $\gamma^1(\cdot; z^1, \tau)$.)

For each $m \in M$, we let $v_1(m)$ and $v_0(m)$ denote the continuation values of having money and not having money, respectively, at the beginning of a period in state m. We use v to denote the vector $[v_0(m), v_1(m)]_{m \in M}$ and we let $\Delta = [\Delta_m]_{m \in M} = [v_1(m) - v_0(m)]_{m \in M}$. Letting $\mathbb{E}^i(\Delta_m; z^i, \tau) = \sum_{n=1}^{N} \gamma^i(m_n; z^i, \tau)\Delta_{m_n}$, the expected value of Δ_m conditional on z^i, the trade under consumer take-it-or-leave-it offers satisfies

$$c(y^i_{z^i}) = \begin{cases} \beta\mathbb{E}^i(\Delta_m; z^i, \tau) \text{ if } c(y^*) \geq \beta\mathbb{E}^i(\Delta_m; z^i, \tau) \\ \\ c(y^*) \text{ otherwise} \end{cases} \tag{3}$$

and

$$\theta^i_{z^i} = \begin{cases} 1 \text{ if } c(y^*) \geq \beta\mathbb{E}^i(\Delta_m; z^i, \tau) \\ \\ c(y^*)/\beta\mathbb{E}^i(\Delta_m; z^i, \tau) \text{ otherwise} \end{cases}, \tag{4}$$

where $y^i_{z^i}$ is output in meeting state z^i and $\theta^i_{z^i}$ is the probability that money is transferred from the consumer to the producer in meeting state z^i. (Under A1, the stationary equilibrium will turn out to be given by the respective first lines of (3) and (4).)

Given a vector of values at the start of the next date, denoted v', and given that trades are determined by (3) and (4), each version of the model gives rise to a Bellman equation denoted $v = f^i(v'; \tau)$, where $f^i(\cdot; \tau) : \mathbb{R}_+^{2N} \to \mathbb{R}_+^{2N}$ and is a function and where, as above, the superscript $i \in \{0, 1\}$ is used to denote the length of the information lag. Notice that $f^0(\cdot; \tau) \neq f^1(\cdot; \tau)$ because and only because $\gamma^0(\cdot; z^0, \tau) \neq \gamma^1(\cdot; z^1, \tau)$. For a given τ, a stationary equilibrium is a fixed point of f^i. The explicit forms of the f^i are given in the appendix. There, it is shown that f^i implies an N-dimensional mapping that involves only the vector $\Delta = [\Delta_m]_{m \in M} = [v_1(m) - v_0(m)]_{m \in M}$, a mapping

$$\Delta = F^i(\Delta'; \tau), \tag{5}$$

where, for a given τ, $F^i : \mathbb{R}_+^N \to \mathbb{R}_+^N$. As is implied by the derivation of F^i, any fixed point of F^i has associated with it a unique fixed point of f^i, and, therefore, a stationary equilibrium. It is slightly more convenient to prove things about the F^i and that is how we proceed.

Our results are for τ's that represent almost perfect persistence, and we start with a lemma about equilibria when $\tau = I$, the $N \times N$ identity matrix, which is perfect persistence.

Lemma 1. The Bellman equation, $\Delta = F^i(\Delta; I)$, has a unique positive solution, denoted $\Delta_m(I)$, which does not depend on i and which satisfies

$$c(y^*)/\beta > \Delta_{m_1}(I) > \Delta_{m_2}(I) > ... > \Delta_{m_N}(I) > 0. \tag{6}$$

The proof appears in the appendix. As might be expected, if $\tau = I$, then the N equations of $\Delta = F^i(\Delta; I)$ are N separate equations, separate in that equation n has only one unknown, Δ_{m_n}. As shown in the proof, that equation is implied by the respective first lines of (3) and (4) and the implied output is the unique positive solution to an equation that has the form

$$c(y) = h(m, \varepsilon)u(y), \tag{7}$$

where $0 < h(m, \varepsilon) < \beta\sigma/(1 - \beta + \beta\sigma)$ (see A1) and where h is decreasing in

m, the inherited state. Those facts give rise to the string of inequalities in the lemma. (Notice that the functions $c(y)$ and $h(m,\varepsilon)u(y)$ are qualitatively similar to those depicted in Figure 1.)

Now, we describe what happens in a neighborhood of $\tau = I$.

Lemma 2. For each version of the model, each $i \in \{0, 1\}$, there exists a neighborhood of $\tau = I$, $\Lambda^i \subset \Pi$ (the set of transition functions on M), such that if $\tau \in \Lambda^i$, then there exists a positive fixed point of $F^i(\cdot, \tau)$, denoted $\Delta^i(\tau)$, which is locally unique and is continuous in τ. Moreover, the associated trades, $y^i_{z^i}(\tau)$, satisfy

$$\lim_{\tau \to I} c(y^1_{z^1}(\tau)) = \beta \Delta_m(I) \text{ for all } z^1 \text{ consistent with } m, \tag{8}$$

and

$$\lim_{\tau \to I} c(y^0_{m'}(\tau)) = \beta \Delta_{m'}(I), \tag{9}$$

where m is the inherited state, and m' is the current state.

The proof is given in the appendix. The first part is an easy application of the implicit function theorem to the equation $0 = F^i(\Delta; \tau) - \Delta$—easy because at $\tau = I$, the m-th component of F^i depends only on Δ_m. That implies that the matrix of partial derivatives, $\partial F^i(\Delta; I)/\partial \Delta$, is diagonal with diagonal entries that are less than one. They are less than one because at the solution for y to (7) the function $c(y)$ crosses the function $h(m, \varepsilon)u(y)$ from below. The limit claims in (8) and (9) play a role in our Phillips-curve result. The first says that the limit of $y^1_{z^1}(\tau)$ does not depend on the whether a trade meeting has *new* or *old* buyers and/or sellers. Given the first part of Lemma 2, it follows from the fact that the posterior distribution, $\gamma^1(\cdot; z^1, \tau)$, converges to placing full weight on the inherited state as $\tau \to I$. In contrast, for any $\tau \in \Lambda^0$, $c(y^0_{m'}(\tau)) = \beta \Delta_{m'}(\tau)$, where m' is the current state.

Now we state and prove our Phillips-curve result. Let $Y^i(m, m'; \tau)$ denote per capita output for version i relative to 2σ, the fraction of single-coincidence meetings. For the information-lag version, we have

$$
\begin{aligned}
Y^1(m, m'; \tau) &= m(1-m)[q_- q_+ y^1_{moo}(\tau) + p_- p_+ y^1_{mnn}(\tau)] + \\
&\quad m^2 q_- p_- y^1_{mon}(\tau) + (1-m)^2 q_+ p_+ y^1_{mno}(\tau),
\end{aligned} \tag{10}
$$

where the suppressed argument of p_+ and p_- is (m, m') and where $q_+ = 1 - p_+$ and $q_- = 1 - p_-$. The four distinct meeting-specific outputs in (10) satisfy (3) for the respective $\gamma^1(\cdot; z^1, \tau)$. For the no information-lag version, we have

$$
Y^0(m, m'; \tau) = m'(1-m') y^0_{m'}(\tau), \tag{11}
$$

where $y^0_{m'}(\tau)$ satisfies (3) for $\gamma^0(\cdot; m', \tau)$.

Proposition. There exists a neighborhood of $\tau = I$, $\Lambda \subset \Pi$, such that if $\tau \in \Lambda$, then (i) $Y^1(m, m'; \tau)$ is strictly increasing in m'; (ii) $Y^1(m, m'; \tau) - Y^0(m, m'; \tau) > 0$ if $m' - m > 0$ and $Y^1(m, m'; \tau) - Y^0(m, m'; \tau) < 0$ if $m' - m < 0$.

Part (i) describes a feature of the information-lag version, while part (ii) compares the two versions. The proof given below relies solely on the continuity results in Lemma 2. The main idea is simple; the same extensive-margin effect, the effect on the measure of trade meetings, is present in both versions, while there is an intensive-margin effect, which works against the existence of a Phillips curve, only in the no information-lag version.

Proof. (i) By (1), the weights in (10) satisfy

$$
m(1-m)(q_- q_+ + p_- p_+) + m^2 q_- p_- + (1-m)^2 q_+ p_+ = m'(1-m'). \tag{12}
$$

When $\tau = I$, output in a trade meeting depends on the state, m, and is the same in both versions of the model. We denote it $\bar{y}_m$, where $c(\bar{y}_m) = \beta \Delta_m(I)$

(see Lemma 1). Using (12) and adding and subtracting $m'(1-m')\bar{y}_m$ to the expression for $Y^1(m, m'; I)$, we get

$$\begin{aligned} Y^1(m, m'; \tau) &= m'(1-m')\bar{y}_m \\ &\quad + m(1-m)[q_- q_+ (y^1_{moo}(\tau) - \bar{y}_m) + p_- p_+ (y^1_{mnn}(\tau) - \bar{y}_m)] \\ &\quad + m^2 q_- p_- (y^1_{mon}(\tau) - \bar{y}_m) + (1-m)^2 [q_+ p_+ (y^1_{mno}(\tau) - \bar{y}_m)]. \end{aligned} \tag{13}$$

By the continuity result in Lemma 2 (see (8)), it follows that

$$\lim_{\tau \to I} Y^1(m, m'; \tau) = m'(1-m')\bar{y}_m, \tag{14}$$

which is increasing in m'.

(ii). By (11) and (13), it follows that

$$\begin{aligned} & Y^1(m, m'; \tau) - Y^0(m, m'; \tau) \\ = \; & m'(1-m')\bar{y}_m - m'(1-m')y^0_{m'}(\tau) \\ & + m(1-m)[q_- q_+ (y^1_{moo}(\tau) - \bar{y}_m) + p_- p_+ (y^1_{mnn}(\tau) - \bar{y}_m)] \\ & + m^2 q_- p_- (y^1_{mon}(\tau) - \bar{y}_m) + (1-m)^2 [q_+ p_+ (y^1_{mno}(\tau) - \bar{y}_m)]. \end{aligned}$$

Therefore, using the limit claims in Lemma 2, we have

$$\lim_{\tau \to I} [Y^1(m, m'; \tau) - Y^0(m, m'; \tau)] = m'(1-m')(\bar{y}_m - \bar{y}_{m'}),$$

and

$$\text{sign}\{\lim_{\tau\to I}[Y^1(m,m';\tau)-Y^0(m,m';\tau)]\} = \text{sign}(\bar{y}_m - \bar{y}_{m'}) = \text{sign}(m'-m),$$

where the last equality follows because $\bar{y}_m = c^{-1}(\beta\Delta_m(I))$ is decreasing in m (see Lemma 1).■

Finally, we should say a word about the price level in the two versions of the model. For each version, we define the price level to be the GDP deflator, per capita nominal output divided by per capita output. Because one unit of money is traded in every trade meeting in both versions of the model, per capita nominal output is the same in both versions and, relative to 2σ, is simply $m'(1-m')$ when the current state is m'. Therefore, if we let $P^i(m,m';\tau)$ denote the price level in version i under transition matrix τ, we have

$$P^i(m,m';\tau) = \frac{m'(1-m')}{Y^i(m,m';\tau)}.$$

It follows from (14) that

$$\lim_{\tau\to I} P^1(m,m';\tau) = \frac{1}{\bar{y}_m},$$

which does not depend on m'. Thus, in the information-lag version and in the neighborhood of $\tau = I$, the response of the price level to the current realization is arbitrarily small. And, because nominal output does not depend on whether or not there is an information lag, we have

$$Y^1(m,m';\tau)P^1(m,m';\tau) = Y^0(m,m';\tau)P^0(m,m';\tau).$$

Therefore,

$$\lim_{\tau\to I} P^0(m,m';\tau) = \frac{1}{\bar{y}_{m'}},$$

which is strictly increasing in m'. Thus, in the no information-lag version and in the neighborhood of $\tau = I$, the price level is strictly increasing in the current stock of money.[5]

4 Our assumptions

We comment here on some of the assumptions that may seem controversial.

4.1 Buyer take-it-or-leave-it offers

We assume buyer take-it-or-leave-it offers for two reasons: it is the optimal trade in the model without any shocks and it is a simple specification. We are confident that our Phillips curve results would also hold for other fixed trading protocols—for example, that the buyer makes a take-it-or-leave-it offer with some positive probability and that the seller makes one with the remaining probability or that the trade maximizes a weighted sum of buyer and seller payoffs provided that some positive weight is given to the buyer payoff.[6]

Given the shocks, our specification may not be optimal from the point of view of ex ante utility given by $m(1-m)[u(y)-c(y)]$. In particular, an ex ante optimum may not have buyer take-it-or-leave-it offers in trade meetings with *new* producers or *new* consumers. Consider new producers who have lost money. The more effort they must expend to reaquire money, the weaker is the incentive to acquire it when they are old producers. Analogous considerations arise for new consumers. Hence, given the shocks, an ex ante optimum may not have buyer take-it-or-leave-it offers in all meetings, even though the objective function is increasing in output in the range of outputs

[5] These results for aggregates are somewhat similar to those obtained by Wallace [13] for a permanent one-time shock. The permanence of the shock in that model seems a bit like perfect persistence, but that model is not a special case of what is presented here. Also, as noted in Wallace [13], there the distinction between new and old traders is ignored, which is valid only when there is no inherited stock in that model.

[6] It is well-known that there is no monetary equilibrium if sellers always make a take-it-or-leave-it offer.

that satisfy the producer's participation constraint.

We did not attempt to study such an optimum for two reasons. It is difficult to say much about the optimum and we view our contribution to be *positive*, not normative. In keeping with that goal, we show that a simple specification can generate a seemingly paradoxical observation—namely, a Phillips curve. Part of that simplicity is that the people in the model are using a simple bargaining protocol.

4.2 Almost perfect persistence

Near perfect persistence seems attractive in the context of the model. To see why, consider two alternative special cases. One is the opposite of perfect persistence, which is unambiguous when there are only two states. With two states, the opposite of perfect persistence is a two-state seasonal and being near it means a small probability that the state remains the same. That seems like a strange specification for aggegate shocks to the stock of money. Less extreme is the special case of *i.i.d.* aggregate shocks. Because the shock in the model determines the stock of money, that also seems strange; it says that this period's stock of money contains no information about next period's stock. And if that seems strange, then so also are specifications of τ near the *i.i.d.* specification. The near perfect-persistence specification keeps us far away from the *i.i.d.* specification.

We admit that we cannot say much about what happens for aggregate shock processes that are not in the neighborhood of perfect persistence. First, even existence of a stationary monetary equilibrium is not straightforward. In particular, as just noted, the function $F^1(\Delta; \tau)$ may not be increasing in Δ under buyer take-it-or-leave-it offers because it is decreasing in the output produced in meetings with new producers and/or new consumers. Therefore, we cannot rely on montonicity, as is done in Katzman et al [4].[7] Second, it is not obvious how to get a general Phillips curve result. While meeting

[7] Presumably, some version of the argument used by Zhu [14] would get us existence.

specific outputs would not depend on the aggregate realization when there is an information lag, those outputs vary with the new/old status of those in a trade meeting, and the weighting over those outputs, which helps determine per capita output, depends on the aggregate shock.

4.3 Individual money holdings in $\{0, 1\}$

Consider the current model, but with individual money holdings in the set $\{0, 1, 2, ..., B\}$ for an arbitrary B, as in Zhu [14]. As is well known, once we depart from $B = 1$, the trades at a date affect the distribution of holdings at the next date, and shocks, even one-time shocks, give rise to transitional dynamics. That greatly complicates the analysis. However, there are a few things we can say.

First, it is easy to model shocks to individuals so that a realization to an individual does not imply an aggregate realization. If a person's pre-shock holding is $x \in \{0, 1, 2, ..., B\}$, it can be assumed that the support of post-shock holdings is some subset of $\{0, 1, 2, ..., B\}$ that does not depend on the current aggregate state, although it can depend on x. Given that condition, trades in meetings with given post-shock holdings of the buyer and seller will be different depending on whether there is an information lag. That is, intensive margin effects will differ for the two versions in a way that is similar to what happens in the $B = 1$ case.

The extensive margin is much more complicated in the richer model. The aggregate shock affects the distribution over kinds of meetings and is not summarized by one number, the measure of trade meetings. And a given aggregate shock can be accomplished by many specifications of individual shocks.

In the $B = 1$ model, we chose parameters (see A1) so that better ex ante outcomes would be achieved if money were replaced by a dividend-bearing asset. That condition could also be imposed in a model with a general bound. If it is imposed, then the best deterministic steady state of the model for a fixed stock of money would almost certainly have a stock which is analogous to a stock of money less than $1/2$ in the $B = 1$ model. In that case, shocks around such an optimum may well give rise to extensive margin effects that are in the same direction as those in the $B = 1$ specification.

4.4 The one-period lag

We adopt the one-period lag only for simplicity. If the lag is lengthened, then, as noted and studied in Araujo and Shevchenko [1] for a one-time shock version, there is asymmetric information in meetings. Also, with a longer lag and on-going shocks, the state of the economy becomes much more complicated. However, there is no reason to expect that the contrast between the no information-lag version and the version with a lag would be qualitatively different from what we have found with a one-period lag. Of course, one interpretation of the one-period lag is that aggregates are revealed at the end of the period.

5 Concluding remark

There are many models that rely on incomplete information to get a Phillips curve. The first was Lucas [6]. But it and most subsequent work that relies on incomplete information do not rely on assumptions that give money a role.[8] The Phillips curve has always been regarded as an association between real and nominal quantities, an association that many, among them Hume, viewed as calling for a special explanation precisely because it relates real and

[8] In Lucas [6], monetary shocks are partially masked by the presence of real shocks, not by dispersed information across markets. (Although the model in Lucas [6] is often called an "island model," all the interesting implications of that model hold in a single island (see Wallace [12]).) Moreover, the incomplete information in that model has nothing to do with the incompleteness needed to give money a role in Lucas's OLG model; namely, that subsequent generations do not observe what the current young do (see Kandori [3] and Kocherlakota [5]).

nominal quantities. Therefore, in order to explain it, it seems desirable to use a model in which money plays a role. We use one of the first such models and are able to show that there is a Phillips curve under mild additional assumptions—the main one being near perfect persistence of the shocks to the stock of money.

6 Appendix

In order to prove Lemmas 1 and 2, we need explicit forms of the functions, f^i and F^i. We begin with the information-lag version.

For each $m \in M$, $v_1(m)$ and $v_0(m)$ denote continuation values at the beginning of a period before the current state is realized and $\Delta_m = v_1(m) - v_0(m)$. Also, in this version, a state of a trade meeting is an element $z \in Z^1 = M \times \{o,n\} \times \{o,n\}$.

In terms of continuation values, the buyer take-it-or-leave-it offer, (y_z, θ_z), is given by (3) and (4), where, by Bayes rule, the posterior $\gamma^1(m; z, \tau)$ is given by

$$\begin{aligned}
\gamma^1(m'; moo, \tau) &= \frac{\tau_{mm'} q_-(m,m') q_+(m,m')}{\sum_{n=1}^N \tau_{mm_n} q_-(m,m_n) q_+(m,m_n)}, \\
\gamma^1(m'; mno, \tau) &= \frac{\tau_{mm'} p_+(m,m') q_-(m,m')}{\sum_{n=1}^N \tau_{mm_n} p_+(m,m_n) q_-(m,m_n)}, \\
\gamma^1(m'; mon, \tau) &= \frac{\tau_{mm'} q_+(m,m') p_-(m,m')}{\sum_{n=1}^N \tau_{mm_n} q_+(m,m_n) p_-(m,m_n)}, \\
\gamma^1(m'; mnn, \tau) &= \frac{\tau_{mm'} p_+(m,m') p_-(m,m')}{\sum_{n=1}^N \tau_{mm_n} p_+(m,m_n) p_-(m,m_n)}.
\end{aligned} \tag{15}$$

In what follows, we use u_z to denote $u(y_z)$ and c_z to denote $c(y_z)$ for each $z \in Z^1$. Let f^1_{jm} for $j \in \{0,1\}$ denote the component of f^1 that maps into

$v_j(m)$. Then, we have

$$f_{jm}^1 = \sum_{m' \in M} \tau_{mm'} v_j(m, m'),$$

where

$$\begin{aligned} v_1(m, m') &= \sigma(1-m)[q_- q_+ u_{moo} - p_- p_+ c_{mnn}] + \sigma m(p_- q_-)[u_{mon} - c_{mon}] \\ & \qquad + \beta\{-\sigma(1-m)(\theta_{moo} q_- q_+ - \theta_{mnn} p_- p_+)\Delta_{m'} - p_- \Delta_{m'} + v_1(m')\}, \end{aligned} \tag{16}$$

and

$$\begin{aligned} v_0(m, m') &= \sigma m[-q_- q_+ c_{moo} + p_+ p_- u_{mnn}] + \sigma(1-m)(p_+ q_+)[u_{mno} - c_{mno}] \\ & \qquad + \beta\{\sigma m(\theta_{moo} q_- q_+ - \theta_{mnn} p_- p_+)\Delta_{m'} + p_+ \Delta_{m'} + v_0(m')\}. \end{aligned} \tag{17}$$

In these expressions, the argument of p_+ and p_-, which is (m, m'), has been suppressed, $q_+ = 1 - p_+$ and $q_- = 1 - p_-$, and $\Delta_{m'} = v_1(m') - v_0(m')$. And, of course, u_z and c_z are given by the buyer take-it-or-leave-it offers as given in (3) and (4), which depend only on the $\Delta_{m'}$. It follows from (16) and (17) that

$$\begin{aligned} F_m^1(\Delta; \tau) &= \sigma \sum_{m' \in M} \tau_{mm'} \{q_- q_+[(1-m)u_{moo} + m c_{moo}] - p_- p_+[m u_{mnn} + (1-m)c_{mnn}]\} \\ & \quad + \sigma \sum_{m' \in M} \tau_{mm'} \{m p_- q_-[u_{mon} - c_{mon}] - (1-m)p_+ q_+[u_{mno} - c_{mno}]\} \\ & \quad + \beta \sum_{m' \in M} \tau_{mm'} [(1 - \sigma\theta_{moo})(1 - p_- - p_+) - \sigma p_+ p_-(\theta_{moo} - \theta_{mnn})] \Delta_{m'}, \end{aligned} \tag{18}$$

where $F^1_m(\Delta;\tau)$ is the m-th component of $F^1(\Delta;\tau)$. Given a fixed point of $F^1(\Delta;\tau)$, the associated fixed point of $f^1(v;\tau)$ can be recovered from (16) and (17).

In an analogous way, we have

$$
\begin{aligned}
F^0_m(\Delta;\tau) &= \sigma \sum_{m'\in M} \tau_{mm'}\{q_- q_+[(1-m)u_{m'} + mc_{m'}] \\
&\qquad +[p_- q_- m - p_+ q_+ (1-m)](u_{m'} - c_{m'})\} \\
&+ \beta(1-\sigma)\sum_{m'\in M} \tau_{m,m'}\theta_{m'}(1-p_- - p_+)\Delta_{m'}. \qquad (19)
\end{aligned}
$$

Now we can prove Lemmas 1 and 2.

Lemma 1. The Bellman equation, $\Delta = F^i(\Delta; I)$, has a unique positive solution, denoted $\Delta_m(I)$, which does not depend on i and which satisfies

$$
c(y^*)/\beta > \Delta_{m_1}(I) > \Delta_{m_2}(I) > ... > \Delta_{m_N}(I) > 0. \qquad (20)
$$

Proof. When $\tau = I$, for each $m \in M$, (18) and (19) are identical and take the simple form

$$
\begin{aligned}
F^i_m(\Delta; I) &= \frac{1-m-\varepsilon}{1-m}\sigma\left\{(1-m)u(y_m) + mc(y_m)\right\} \\
&\qquad +\beta(1-\sigma)\frac{1-m-\varepsilon}{1-m}\theta_m\Delta_m,
\end{aligned} \qquad (21)
$$

where, in this special case, (y_m, θ_m) is given by (3) and (4) and depends only on Δ_m.

Fix some $m \in M$. For any given $(y_m, \theta_m) \in \mathbb{R}_+ \times [0,1]$, by (21), any Δ_m satisfies $\Delta_m = F^i_m(\Delta; I)$ if and only if (y_m, θ_m) satsifies (3) and (4) for Δ_m, and

$$\Delta_m = \frac{\frac{1-m-\varepsilon}{1-m}\sigma\left[(1-m)u(y_m)+mc(y_m)\right]}{1-\theta_m\beta(1-\sigma)\frac{1-m-\varepsilon}{1-m}}. \tag{22}$$

Now we show that $F_m^i(\cdot; I)$ has a unique positive fixed point and that for that fixed point, the implied $\theta_m = 1$. Since the right-hand side of (22) is increasing in θ_m, it suffices to show that when $\theta_m = 1$, there is a unique fixed point for which the implied $y_m < y^*$. So take $\theta_m = 1$. By the first line of (3), $\Delta_m = c(y_m)/\beta$. Substituting that into (22), we get

$$c(y_m) = h(m,\varepsilon)u(y_m), \tag{23}$$

where

$$h(m,\varepsilon) = \frac{\beta\sigma(1-m-\varepsilon)}{\beta\sigma(1-m-\varepsilon)+1-\frac{1-m-\varepsilon}{1-m}\beta} = \frac{\beta\sigma}{\beta\sigma+\frac{(1-\beta)(1-m)+\varepsilon\beta}{(1-m-\varepsilon)}}.$$

Because $\frac{(1-\beta)(1-m)+\varepsilon\beta}{(1-m-\varepsilon)} > (1-\beta)$, $h(m,\varepsilon) < \frac{\beta\sigma}{\beta\sigma+(1-\beta)}$. Therefore, by the Inada condition, $u'(0) = \infty$ or $c'(0) = 0$, there exists a unique positive solution to (23), denoted $\bar{y}_m$, and, by the last inequality and A1, $\bar{y}_m < y^*$. This, as mentioned earlier, implies that there is no fixed point with $\theta_m < 1$. It also gives us the first inequality in (20). Also, because $\frac{(1-\beta)(1-m)+\varepsilon\beta}{(1-m-\varepsilon)}$ is increasing in m, $h(m,\varepsilon)$ is decreasing in m. That implies the string of inequalities in (20). Finally, because the function $c(y_m)$ crosses the function $h(m,\varepsilon)u(y_m)$ from below at $\bar{y}_m$, it follows that

$$c'(\bar{y}_m) > h(m,\varepsilon)u'(\bar{y}_m), \tag{24}$$

a fact we use below.■

Lemma 2. For each version of the model, each $i \in \{0,1\}$, there exists a neighborhood of $\tau = I$, $\Lambda^i \subset \Pi$, such that if $\tau \in \Lambda^i$, then there exists a positive fixed point of $F^i(\cdot,\tau)$, denoted $\Delta^i(\tau)$, which is locally unique and is continuous in τ. Moreover, the associated trades, $y^i_{z^i}(\tau)$, satisfy

$$\lim_{\tau \to I} c(y^1_{z^1}(\tau)) = \beta \Delta_m(I) \text{ for all } z^1 \text{ consistent with } m,$$

and

$$\lim_{\tau \to I} c(y^0_{m'}(\tau)) = \beta \Delta_{m'}(I),$$

where m is the inherited state, and m' is the current state.

Proof. First we show that the function $F^1(\cdot;\tau)$ has a unique positive fixed point in a neighborhood of $(\Delta(I), I)$, where $\Delta(I)$ is the Lemma-1 fixed point. Here we treat τ as a vector in the space $\mathbb{R}^{N \times N}$. Although τ is not a transition matrix everywhere in such a neighborhood, the function $F^1(\Delta;\tau)$ is well-defined everywhere in such a neighborhood. Moreover, it is continuously differentiable. In particular, because $\bar{y}_m < y^*$ for all m in a neighborhood of $(\Delta(I), I)$, the trades are determined by a binding producer constraint, the respective first lines of (3) and (4). This guarantees differentiability and allows us to use the implicit function theorem.

We start by displaying the partial derivatives of $F^1_m(\Delta;\tau)$ in (21) evaluated at $(\Delta, \tau) = (\Delta(I), I)$ (note that we have $\theta_m = 1$ at that point). We have $\partial F^1_m(\Delta(I), I)/\partial \Delta_{m'} = 0$ if $m' \neq m$ and

$$\frac{\partial F^1_m(\Delta(I); I)}{\partial \Delta_m} = \beta \frac{1 - m - \varepsilon}{1 - m} \left\{ \sigma \left[(1 - m) \frac{u'(\bar{y}_m)}{c'(\bar{y}_m)} + m \right] + (1 - \sigma) \right\}.$$

Then using (24) and letting $x = 1 - m - \varepsilon$, we have

$$
\begin{aligned}
\frac{\partial F^1_m(\Delta(I);I)}{\partial \Delta_m} &< \beta \frac{x}{1-m} \left\{ \sigma \left[\frac{(1-m)}{h(m,\varepsilon)} + m \right] + (1-\sigma) \right\} \\
&= \beta \frac{x}{1-m} \sigma \left\{ (1-m) \frac{\beta\sigma x + 1 - \frac{x}{1-m}\beta}{\beta \sigma x} + m \right\} + \beta \frac{x}{1-m}(1-\sigma) \\
&= \beta\sigma x + 1 - \frac{x}{1-m}\beta + \beta\frac{x}{1-m}\sigma m + \beta \frac{x}{1-m}(1-\sigma) \\
&= \beta x \left\{ \sigma - \frac{1}{1-m} + \frac{\sigma m}{1-m} + \frac{1-\sigma}{1-m} \right\} + 1 \\
&= \beta x \frac{1}{1-m} \left\{ (1-m)\sigma - 1 + \sigma m + 1 - \sigma \right\} + 1 = 1.
\end{aligned}
$$

That is, the matrix, $\partial F^1(\Delta(I), I)/\partial \Delta$ is a diagonal matrix with entries on the diagonal that are less than one. Therefore, $\partial[F^1(\Delta(I);I) - \Delta(I)]/\partial\Delta$ has full rank. Thus, by the implicit function theorem, there is a neighborhood of (Δ^1, I), denoted Γ^1, such that for all $\tau \in \mathrm{Proj}_\tau \Gamma^1$, there is a unique $\Delta^1(\tau)$ such that $F^1(\Delta^1(\tau);\tau) - \Delta^1(\tau) = 0$ and $\Delta^1(\tau)$ is continuous in that neighborhood. Let $\Lambda^1 = (\mathrm{Proj}_\tau \Gamma^1) \cap \Pi$. The arguments for F^0 are exactly the same because $\partial F^1_m(\Delta(I);I)/\partial\Delta_{m'} = \partial F^0_m(\Delta(I);I)/\partial\Delta_{m'}$ for all m, m'.

Finally, the claim about the limit of $c(y^1_z(\tau))$ follows from the first line of (3) and two distinct limit conclusions: one is that $\lim_{\tau\to I}\Delta^1(\tau) = \Delta(I)$ and the other is that for any $z \in Z^1$ consistent with m, $\lim_{\tau \to I}\gamma^1(m'; z, \tau) = 1$ if $m' = m$ (and zero otherwise). The first follows from the first part of Lemma 1, while the second follows directly from the expressions for the $\gamma^1(m'; z, \tau)$ in (15). The claim about the limit of $c(y^0_{m'}(\tau))$ follows directly from (3) and $\lim_{\tau\to I}\Delta^0(\tau) = \Delta(I)$. ■

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