Unique Monetary Equilibrium with Inflation in a Stationary Bewley-Aiyagari Model

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Abstract

We prove the existence and uniqueness of a stationary monetary equilibrium in a Bewley-Aiyagari model with idiosyncratic shocks. This is an exchange economy with an infinite horizon and one consumption good, and with each agent facing idiosyncratic endowment shocks at each period; the agents may trade their endowments for the only asset, fiat money. The government increases the money supply at a constant growth rate that induces inflation in a stationary monetary equilibrium. We identify the necessary and sufficient condition for a stationary monetary equilibrium (where money has a positive value and the aggregate real balance is constant over time) to exist, and, when it exists, we show that it is unique. The argument for uniqueness is based on a new monotonicity result for the average optimal consumption.

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1 Introduction

The Bewley-Aiyagari (Bewley [4], Aiyagari [3], and Hugget [8]) model is the workhouse for modern dynamic macroeconomic models. In these models, each agent is subject to idiosyncratic income shocks and trades in a competitive spot market for certain assets under some borrowing constraints. While equilibrium existence has been established in various contexts (see, for example, Acemoglu and Jensen [1]), as discussed in Miao [17],[1] there are no general theoretical results for uniqueness of a stationary equilibrium in this type of model. Uniqueness results require some sort of monotonicity in policy functions, which can be difficult to obtain for the underlying dynamic programming problem. In the Bewley-Aiyagari model with capital accumulation, multiple equilibria cannot be excluded in general. As argued in Acemoglu and Jensen [1], such multiplicity can be problematic for numerical methods and our understanding on how the economy would respond to changes in the fundamentals.

In this paper we study a pure-currency economy in the Bewley-Aiyagari model. This model and its variants have been used to study inflation caused by lump-sum injections of money (see Imrohoroglu [9], Green and Zhou [7], and Wallace [22]). In our model there is a single perishable consumption good, and there is a population of infinitely lived agents who face idiosyncratic endowment shocks at each period. At each period there is a competitive market where agents trade between their endowments and the only asset, money, which is intrinsically useless but is perfectly durable and recognizable. Lack of commitment and enforcement implies that there are no (incentive) feasible intertemporal arrangements. Against this background environment, the government increases the money supply each period at a constant growth rate and distributes it equally to all agents in a lump-sum manner.

We focus on stationary monetary equilibria in this model. In contrast to similar models with capital accumulation in which the (rental) price of capital is determined by current

[1] See comments after Definition 17.1.1.

aggregate capital holdings, in our economy the price of money also depends on its future prices and hence can only be determined endogenously. In particular, there is always an equilibrium under which money is not valued, and an equilibrium is *monetary* only if the equilibrium price of money is strictly positive. Stationarity has two requirements: first, the total real value of money (or *real balance*) is constant over time (and, hence, when money supply grows at a positive rate, there is inflation); second, the distribution of money holdings is constant over time. We give the necessary and sufficient condition for such an equilibrium to exist. More importantly, we show that when it exists, it is also unique.

In this environment, an agent's optimal money holding depends on his endowment shock and his previous money holding, and hence follows a first-order Markov process. Under a constant money supply, the stationary monetary equilibrium is determined by the unique invariant distribution of this Markov process. With a positive growth rate of the money supply, however, this is no longer the case, as the real value of the lump-sum transfer of money is also an equilibrium object, and market clearing gives an equilibrium condition that relates the average consumption under the equilibrium distribution and the real value of the lump-sum transfer. We prove uniqueness by showing that the average consumption is monotonic in the real value of the lump-sum transfer. This monotonicity result is new and our arguments rely on the ergodic theorem, which relates the long-run average of consumptions from an "typical" individual realization of endowment shocks to the average consumption of the stationary cross-section distribution of consumption. While results similar to our existence results have been established in earlier papers, our uniqueness result is new.

Our uniqueness results implies real determinacy in a monetary model, at least among stationary monetary equilibria. This is in contrast with the indeterminacy result in Green and Zhou [6], which shows the existence of a continuum of stationary monetary equilibria in the context of a Kiyotaki-Wright [10] model but with divisible money holdings and indivisible goods, and under double auctions. Moreover, since the unique equilibrium is upper hemi-continuous in the money creation rate, the equilibrium *allocation* is also continuous. This implies that an optimal money creation rate (and hence the implied inflation rate) exists, and the literature has provided examples in which such a rate is strictly positive, such as Green and Zhou [7].[2] In contrast with the Lagos-Wright [13] model, in which the unique stationary monetary equilibrium features a degenerate distribution of money holdings, but inflation through lump-sum transfers is never optimal, in our model a monetary equilibrium exists only if the equilibrium distribution has a non-degenerate support and inflation can be optimal (see Wright [23] for the uniqueness of stationary monetary equilibria under lump-sum injections of money in the context of the Lagos-Wright [13] model).

More broadly speaking, our paper contributes to a recent literature on the existence and uniqueness of equilibria in Bewley-Aiyagari models. Açikgöz [2] proves the existence of a stationary equilibrium in such a model with capital accumulation, but, in contrast to our pure-currency economy, he also demonstrates that there can be multiple stationary equilibria. As pointed out there, one reason for this multiplicity is the income effects of a higher rate-of-returns on assets, and Lehrer and Light [14] give a sufficient condition on the underlying utility function for the substitution effect to dominate the income effect; Light [15] uses that condition to obtain uniqueness.

Finally, our uniqueness result allows for unambiguous comparative statics, and one can apply existing comparative-statics results to our setup. In particular, although our setup is not a special case of that considered in Acemoglu and Jensen [1], we can readily translate their results in our setup. For example, given their results, it is easy to show that an increase in the discount factor will lead to an increase in the equilibrium real

[2]There are also other examples which show that inflation can be socially beneficial in closely related models. Rocheteau, Weill, and Wong [18], [19], based on variants of the Lagos-Wright [13] model, study schemes that result in inflation but may involve interest payments on money holdings (as opposed to lump-sum transfers) and can be welfare improving. Lippi, Ragni, and Trachter [16], based on the Scheinkman-Weiss [20] model, study optimal inflation responding to aggregate shocks.

balances in our setup.

2 Model

2.1 Environment

Time is discrete and there is an infinite horizon. There is a population of agents who are ex ante identical and there is one perishable good. Let $u(c)$ be the agent's utility from consuming $c \geq 0$ units of the good. We assume that $u : \mathbf{R}_+ \to [0, \bar{u}]$, where $u(0) = 0$ is bounded,[3] strictly increasing and strictly concave. Agents maximize discounted expected utility with discount factor $\beta \in (0, 1)$. At the beginning of each period each agent receives an idiosyncratic shock to his endowment, denoted by y. We assume that y is drawn from a closed interval $Y = [\underline{y}, \overline{y}]$ with $0 \leq \underline{y} < \overline{y}$ and is i.i.d. across periods from distribution $\pi \in \Delta(Y)$ such that $\underline{y}, \overline{y} \in \text{support}(\pi)$. Agents cannot commit to future actions and there is no record-keeping technology for credit arrangements. There is an intrinsically useless asset called money, and in each period agents may trade their money holdings against the consumption good in a competitive market.

We assume that the government increases the money supply at a constant net growth rate $\gamma \geq 0$, and that the newly created money is distributed to agents as a lump-sum transfer. It is convenient to normalize the agent's money holdings as fractions of the average money holding. With this normalization, lump-sum distribution of money with constant growth rate $\gamma \geq 0$ is equivalent to a proportional taxation on money holdings with a lump-sum transfer: an agent who holds m units of money before the transfer will end up with $(m + \gamma)/(1 + \gamma) = (1 - \tau)m + \tau$ units after the transfer, where $\tau = \gamma/(1 + \gamma)$. We call τ the *tax rate on money holding*. Thus, the course of action in each period has three stages: first, agents receive their endowments; second, agents trade between their

[3] We assume a bounded utility function to avoid technical issues with the individual dynamic programming problem (see Kuhn [12] and references therein for issues regarding unbounded utility functions). However, our argument for the uniqueness of the stationary monetary equilibrium does not depend on that.

endowments and money; finally, the tax on money holding and transfer occurs.

Fix a sequence $p_0, p_1, p_2, \dots \in \mathbf{R}_+$ of *prices of money in terms of goods*. Consider a single agent who maximizes expected discounted utility assuming the price of money in terms of goods at every period t is p_t. The agent's problem is a dynamic optimization problem with state variable (m, y) representing money holding and endowment before the trade. Let $V_t(m, y)$ be the optimal continuation value under state (m, y) at period t. Compressing the dependence on $p_0, p_1, \dots$, the Bellman Equation for V_t is given by

$$V_t(m,y) = \max\left\{u(c) + \beta \int V_{t+1}\left((1-\tau)m' + \tau, y'\right)\ \pi(\mathrm{d}y') : 0 \le c, m',\ c + m'p_t \le y + mp_t\right\} \tag{1}$$

for every $t \geq 0$. Here the choice variables c and m' are the consumption and post-trade money holding.

Definition 1. For a given initial distribution of money holdings, $\mu_0 \in \Delta\big(\mathbf{R}_+\big)$, an **equilibrium with a tax rate on money holding** τ is a sequence $p_0, p_1, \dots$ of prices of money in terms of goods, and a sequence $\mu_1, \mu_2, \dots \in \Delta\big(\mathbf{R}_+\big)$ of distributions of money holdings with $\int m\ \mu_t(\mathrm{d}m) = 1$ for all $t \geq 0$, such that the following holds for every $t \geq 0$:[4]

1. Law of motion: $\mu_{t+1}(A) = \mu_t \otimes \pi\left(\{(m,y) : (1-\tau)\phi_t(m,y) + \tau) \in A\}\right)$ for every Borel subset A of $\mathbf{R}$.

2. Market clearing: $\int c_t(m,y)\ \mu(\mathrm{d}m)\ \pi(\mathrm{d}y) = \int y\ \pi(\mathrm{d}y)$.

Here c_t and ϕ_t are the optimal consumption and post-trade monetary holding that correspond to the sequence $p_0, p_1, \dots$ of prices. The equilibrium is *monetary* if $p_t > 0$ for every $t \geq 0$.

The law of motion reflects the fact that if the distribution of money holdings at the beginning of period t is μ_t and agents trade optimally then the distribution of money

[4] We use $\mu_t \otimes \pi$ to denote the joint distribution of money holdings and endowments under independence of the two

holdings at the beginning of period $t+1$ is given by μ_{t+1}. The market-clearing conditions express the clearing of the market for the consumption good at each date. By Warlas' Law there is a sequence of equivalent conditions in terms of money holdings.

It is easy to verify that there is a nonmonetary equilibrium under which $p_t = 0$ for all $t \geq 0$. In this equilibrium money has no value and all agents consume their daily endowments. In contrast, we are interested in a monetary equilibrium under which money has a positive value.

While our setup is closely related to that in Acemoglu and Jensen [1], there are two differences. First, in their setup each agent solves a dynamic programming problem that depends on some parameter (that corresponds to p_t in our model), which, under market clearing, is determined by an exogenous function of the aggregation of individual choices. In contrast, the sequence $\{p_t\}_{t=0}^{\infty}$ in our setup is endogenous and the market clearing conditions relate the aggregation of individual choice to the aggregation of the endowments. Second, our interest in monetary equilibrium has no counterpart in their environment. For these reasons we cannot directly use their results in our setup. In particular, the question for which initial distribution of money holdings μ_0 there exists some monetary equilibrium is an open question in our setup.

Remark 1. The conditions in Definitions 1 can be interpreted in two ways: from the perspective of a single agent, and from the perspective of the population. Consider first a single agent whose initial money holding is randomized from μ_0, who receives a stochastic stream of shocks, and who consumes and saves according to c_t and ϕ_t. The time series of consumption and money holding of the agent is then a stochastic process. The market-clearing conditions imply that μ_t is the distribution of money holding at day t and that the expectation of date t's consumption equals the expected endowment. At the population level, the dynamic is completely deterministic: μ_t is the empirical distribution of money holdings at the beginning of period t and $\mu_t \otimes \pi$ is the empirical joint distribution of money holdings and endowments. More explicitly, we can identify the set of agent with

$\mathbf{R}_+ \times [\underline{y}, \overline{y}]^{\mathbb{N}}$ equipped with a measure $\mu_0 \otimes \pi^{\otimes \mathbb{N}}$, so that each agent is identified with his initial money holding and the infinite sequence of endowments. From this population perspective, the market-clearing conditions say that the population consumes the total endowment. This dual perspective is a basic feature of the Bewley-Aiyagari models. We use the stochastic single-agent perspective to derive the optimal consumption and saving strategy of the agents, and we use the deterministic population perspective to describe what will actually happen.[5]

We focus on the stationary monetary equilibrium defined below.

Definition 2. An equilibrium with a tax rate on money holding τ is *stationary* if $p = p_0 = p_1 = \dots$ and $\mu = \mu_0 = \mu_1 = \dots$ for some $p \in \mathbf{R}_+$ and $\mu \in \Delta(\mathbf{R}_+)$ with $\int m \ \mu(\mathrm{d}m) = 1$. The stationary equilibrium is monetary if $p > 0$.

In a stationary monetary equilibrium with tax rate τ, the price for money is constant over time. However, remember that this is because we normalized the average money supply to one unit per agent, and hence, the price for money before the normalization (the version where the money supply increases at rate γ per period) decreases at a constant rate. This then implies that there is a constant inflation at rate γ.

Note that under a stationary equilibrium the stochastic process that represents the consumption and saving of a single agent (as mentioned in Remark 1) is stationary. By the ergodic theorem, the expectation of daily consumption also equals the long run realized daily consumption. Therefore, under a stationary equilibrium, the realized (across-period) average consumption of the agent equals the mean endowment.

Now we are ready to present our main result.

Theorem 1. *There exists a stationary monetary equilibrium if and only if*

$$u'(\overline{y}) < \beta(1-\tau) \cdot \int u'(y) \ \pi(\mathrm{d}y). \tag{2}$$

[5]Readers who would like us to embed the two perspectives in a single model, with stochastic processes for each agent and appeal to some "exact law of large numbers" in order to justify "no aggregate uncertainty" are referred to Section II.B in Acemoglu and Jensen [1] and the references therein.

When it exists, it is also unique and the invariant distribution μ of money holding is non-degenerated.

Theorem 1 gives the precise condition for a stationary monetary equilibrium to exist, and, when it exists, states that it is unique. The condition (2) has several requirements for such an equilibrium to exist. First, the inflation rate cannot be too high. Second, the endowment shocks must be sufficiently diverse. Third, the discount factor should be sufficiently high. Finally, the utility function should be sufficiently concave. When condition (2) fails, it is optimal for each agent to stay in autarky, and hence, money has no value in equilibrium.

Uniqueness allows for a unambiguous comparative statics, and one may directly apply some known results. In particular, although our setup is not a special case of that considered in Acemoglu and Jensen [1], it is easy to show that an increase in any "positive shocks" defined there (i.e., any changes in exogenous parameters that will lead to an increase in the policy function ϕ) will lead to an increase in the equilibrium real balances in our setup. One such shock is an increase in the discount factor β. However, an increase in the endowment (in the first-order-stochastic-dominance sense) is not a positive shock in general, as it has two opposing effects to the optimal money holding: while it gives more resources to save today, it also gives a better future so that there is less need to save today (the income effect).

The existence part of Theorem 1, though apparently new, follows from standard arguments. Geanakoplos et al. [5] also give an existence result in a closely related economy with inflation. However, because a cash-in-advance constraint is assumed there, there is no need for any condition analogous to (2). Our main contribution is to show that such an equilibrium is unique.

3 Proof of Theorem 1

After some mathematical preliminaries (Section 3.1), we first write the Bellman equation for the single-agent problem and the definition of stationary equilibrium in terms of real values (Definition 3 in Section 3.2). With this formulation, the state space represents the real wealth of the agent at every period after the endowment shock.

The argument for proving existence is standard. Since, for each individual agent, the optimal real balances across periods follow a Markov process, we show that that process satisfies the mixing condition in Stokey and Lucas [21] and hence has a unique ergodic distribution. When $\tau = 0$, that ergodic distribution fully describes the unique stationary monetary equilibrium. When $\tau > 0$, however, a fixed-point argument is needed to establish the existence, because the single-agent problem depends on the lump-sum transfers in real terms, denoted by $b = p\tau$, which in turn depend on the equilibrium average holding of real balances p. To prove uniqueness of the fixed point when $\tau > 0$, we need some sort of monotonicity with respect to b. The core of our argument is that, while the individual policy function may not be monotonic, we can show that the average optimal consumption is increasing in b. We give this monotonicity result in Section 3.3, where we study the individual dynamic programming problem. Finally, we use the monotonicity result to prove Theorem 1 in Section 3.4.

3.1 Preliminaries

3.1.1 Notations

If $\mu, \nu \in \Delta(\mathbf{R})$ are probability distributions and $f : \mathbf{R} \to \mathbf{R}$, we denote by $f(\mu) \in \Delta(\mathbf{R})$ the push-forward of μ under f. This is the distribution of $f(X)$, where X is a random variable with distribution μ. In the special case that $f(x) = ax + b$, we also denote $f(\mu) = a\mu + b$. We denote by $\mu * \nu$ the convolution of μ and ν. This is the distribution of $X + Y$, where X, Y are independent random variables with distributions X and Y.

Monotone markov chains

Let $(W, \mathcal{W})$ be a Borel space. A *transition probability* over W is a function $\chi : W \times \mathcal{W} \to [0, 1]$ such that $\chi(w, \cdot)$ is a probability distribution over $(W, \mathcal{W})$ for every $w \in W$ and $\chi(\cdot, A)$ is a Borel function for every $A \in \mathcal{W}$. We let $T : \Delta(W) \to \Delta(W)$ be the *stochastic operator of* χ given by $T(\lambda)(A) = \int \chi(w, A)\lambda(\mathrm{d}w)$ for every $A \in \mathcal{W}$.

Fix a transition probability χ over $(W, \mathcal{W})$. For every $\lambda \in \Delta(W)$, we denote by P_λ the distribution of a sequence $X_0, X_1, \ldots$ of W-valued random variables such that

$$\begin{aligned} &X_0 \sim \lambda, \text{ and} \\ &P_\lambda(X_{k+1} \in \cdot | X_0, \ldots, X_k) = \chi(X_k, \cdot) \text{ for every } k \geq 0. \end{aligned} \tag{3}$$

When $\lambda = \delta_w$ is the Dirac measure over $w \in W$, we also denote $P_\lambda = P_w$. A probability distribution λ over W is called an *invariant distribution of* χ if $T\lambda = \lambda$. Equivalently λ is an invariant distribution if the stochastic process $X_0, X_1, \ldots$ given in (3) is stationary. For every $B \in \mathcal{W}$, let T_B be the (possibly infinite) first time in which the process hits B:

$$T_B = \inf\{1 \leq t : X_t \in B\}.$$

The invariant distribution λ is *ergodic* if $\mathbf{P}(T_B < \infty) = 1$ for every $B \in \mathcal{W}$ such that $\lambda(B) > 0$. The fundamental feature of ergodic distributions, captured by the Ergodic Theorem [17, Theorem 4.3.2], is that the expected value of a function $f : W \to \mathbf{R}$ of the state of the process at any given time almost surely equals the time average of the function of the sample path of the process.

For our main argument we will use the following lemma (Kac's Lemma, c.f. Krengel [11], Proposition 6.8): The expectation of a function f of the state in any given time equals the expected average of the function between two entries to a set B. The lemma is an immediate consequence of the ergodic theorem, but in fact it follows from more basic principles.

Lemma 1. *Let χ be a transition probability over $(W, \mathcal{W})$, and let λ be an ergodic invariant distribution. Let $f : W \to \mathbf{R}$ be Borel measurable and bounded. Then for every $B \in \mathcal{W}$ such that $\lambda(B) > 0$, it holds that*

$$\mathbb{E}f(X_0) = \mathbb{E}\left(1_{\{X_0 \in B\}} \sum_{t=0}^{T_B - 1} f(X_t)\right),$$

where $X_0, X_1, \ldots$ are the W-valued random variables given by (3).

The transition probability χ is *uniquely ergodic* if it admits a unique invariant distribution λ. If χ is a uniquely ergodic transition probability, then its invariant distribution is ergodic.

We assume from now on that $W = [\underline{w}, \overline{w}] \subseteq \mathbf{R}^n$ is an n-dimensional interval, equipped with the standard compact lattice structure.

The transition probability χ is *monotone* if $\chi(w, \cdot) \leq_{st} \chi(w', \cdot)$ whenever $w \leq w'$. Equivalently, χ is monotone if the corresponding stochastic operator $T : \Delta(W) \to \Delta(W)$ is monotone in first-order stochastic dominance. We present two propositions, Propositions 1 and 2, which are taken from Stokey and Lucas [21].[6]

Proposition 1. *Every monotone transition probability admits an invariant distribution.*

Proposition 2. *Let χ be a monotone transition probability. If there exists some $w \in \mathbf{R}^n$ such that $\underline{w} < w < \overline{w}$ and such that*

$$P_{\overline{w}}\left(X_k < w \textit{ for some } k \geq 0\right) > 0, \textit{ and } P_{\underline{w}}\left(X_k > w \textit{ for some } k \geq 0\right) > 0, \tag{4}$$

then χ is uniquely ergodic. Moreover, if (4) *holds for every such w, then $\underline{w}, \overline{w}$ are in the support of the invariant distribution of χ.*

[6] The assertion about the support in Proposition 2 does not seem to be stated in Stokey and Lucas [21], but it follows immediately from the proof.

3.2 Equivalent definition of stationary equilibrium

Section 2 provided a definition of a stationary monetary equilibrium. For our proof, however, it is convenient to view each consumer problem as a parametrized dynamic programming with parameter space $\mathbf{R}_+$ representing government transfer b and state space $\mathbf{R}_+$ representing the wealth $w = pm + y$ of the agent at every period (after the endowment shock and before the trade), both expressed in real terms. Let $V_b(w)$ be the corresponding value function, and the Bellman equation for $V_b(w)$ becomes

$$V_b(w) = \sup_{0 \le c \le w} \left\{ u(c) + \beta \int_{y \in Y} V_b\big((1-\tau)(w-c) + y + b\big)\ \pi(\mathrm{d}y) \right\}. \tag{5}$$

By standard dynamic programming arguments it follows from the assumptions on u that the supremum is achieved at a unique consumption level. We denote by $c_b(w)$ the optimal consumption and $s_b(w) = w - c_b(w)$ the optimal real balance holding at the end of a period. Now, we may rewrite the equilibrium objects in Definition 2 as follows.

Definition 3. A *stationary monetary equilibrium with tax rate on money holding* τ is given by a distribution $\lambda \in \Delta\big(\mathbf{R}_+\big)$ of wealth and a real government transfer b, such that

1. Invariance: $\lambda = (1-\tau)s_b(\lambda) * \pi + b$.

2. Government balance: $b = \tau \int s_b \ \mathrm{d}\lambda$.

3. Positive real balances: $\int s_b \ \mathrm{d}\lambda > 0$.

The relationship between the equilibrium objects (μ, p) and (λ, b) in Definitions 2 and 3 is given by

$$p = \int s_b \ \mathrm{d}\lambda, \quad \mu = \left((1-\tau)s_b(\lambda) + b\right)/p, \text{ and}$$
$$b = \tau p, \quad \lambda = p\mu * \pi.$$

Definition 3 highlights the additional difficulty that is involved when $\tau > 0$. Indeed, for $\tau = 0$ the government-balance condition implies that $b = 0$. In this case, as we shall see, the existence and uniqueness of a stationary monetary equilibrium follows immediately from the existence and uniqueness of the invariant distribution of the Markov transition induced by the random endowment and the agent's optimal saving. For $\tau > 0$ we need to find the pair b and λ that satisfies the invariance condition and the government-balance condition simultaneously. This requires an appeal to some fixed-point argument (which is relatively easy, because b is a one-dimensional entity). Uniqueness requires some monotonicity result, which is the main theoretical contribution of this paper.

3.3 The individual consumption saving problem

In this subsection we study the single consumer's problem (5). By standard dynamic programming arguments, it follows from the assumptions on u that V_b is bounded, continuous, monotone increasing, submodular, and strictly concave in w; that the supremum is achieved at a unique consumption level and that the optimal consumption $c_b(w)$ is continuous and strictly increasing in b, w, and $\lim_{w\to\infty} c_b(w) = \infty$; that the optimal saving $s_b(w) = w - c_b(w)$ is strictly decreasing in b and strictly increasing in w; that *Euler's equation*

$$u'(c_b(w)) \geq \beta(1-\tau) \int u'(c_b[(1-\tau)s_b(w) + y + b])\ \pi(\mathrm{d}y) \tag{6}$$

is satisfied, with equality if $c_b(w) < w$; and that $c_b(\cdot)$ is the unique function that satisfies Euler's equation (6) and the transversality condition.

Let

$$\eta_b(w, y) = (1-\tau)s_b(w) + y + b \tag{7}$$

be the next-period wealth of an agent who has wealth w at the current period, and gets the lump-sum transfer b and endowment shock y next period. The following claim summarizes properties of the next-period wealth function.

Claim 1. *Let $\eta_b(w, y)$ be the next period wealth function given by (7). Then $\eta_b(w, y)$ is continuous, monotone increasing in w and y, and $\eta_b(w, y) - w$ is strictly decreasing in w*

and increasing in y. Moreover, $\eta_b(w, \bar{y}) - w < 0$ for a sufficiently large w.

Proof. The first assertion follows from (7) and the corresponding properties of the optimal saving function. The second assertion follows from $\eta_b(w, y) - w = -c_b(w) - \tau s_b(w) + y + b$ and monotonicity of the optimal consumption and saving functions. The last assertion follows from the fact that $\lim_{w\to\infty} c_b(w) = \infty$. $\square$

Invariant distribution

By Claim 1, there exists a unique $\overline{w}_b \in \mathbf{R}_+$ such that $\eta_b(\overline{w}_b, \overline{y}) = \overline{w}_b$; let $\underline{w}_b = b + \underline{y}$ so that $\eta_b(\underline{w}_b, \underline{y}) = \underline{w}_b$ since $c_b(b + \underline{y}) = b + \underline{y}$ from the Euler's equation. Consider the transition probability χ_b on $W = [\underline{w}_b, \overline{w}_b]$ such that $\chi_b(\cdot|w) = \eta_b(w, \pi)$. The corresponding stochastic operator is given by

$$T_b(\lambda) = (1 - \tau)s_b(\lambda) * \pi + b. \tag{8}$$

Then it follows from Claim 1 and the definition of $\underline{w}_b, \overline{w}_b$ that χ_b is a monotone transition on $[\underline{w}_b, \overline{w}_b]$. The following claim and its proof are standard.

Claim 2. *For every $b \geq 0$, the transition χ_b is uniquely ergodic. Moreover, if λ_b is the invariant distribution then $\underline{w}_b, \overline{w}_b \in$ support(λ_b).*

Proof. Using Proposition 2 we need to show that (4) holds for every w such that $\underline{w}_b < w < \overline{w}_b$. Indeed, let $\underline{y} < y$ be sufficiently small such that $\eta_b(w, y) < w$, with existence following from the definition of $\underline{w}_b$ and Claim 1. Then it follows from Claim 1 that $\eta_b(w', z) < w$ whenever $w' < w$ and $\underline{y} \leq z \leq y$, and that $\eta_b(w', z) < w'$ whenever $w' \geq w$ and $\underline{y} \leq z \leq y$. These properties and the continuity of η imply that there exists some N such that, for every sequence, $z_1, \ldots, z_N$, of endowments such that $z_i \in [\underline{y}, y]$, receiving these endowments in consecutive periods will decrease the consumer wealth to below w, regardless of the initial wealth. Since there is a positive probability that the endowment of the agent at a given period is in $[\underline{y}, y]$, the first condition of Proposition 2 is indeed satisfied. The second condition is proved analogously. $\square$

Monotonicity

The following lemma is the core of the proof. It says that agents who receive more government transfer will consume more on average.

Lemma 2. *Let λ_b be the unique ergodic distribution given by Claim 2. The function $b \mapsto \int c_b \, \mathrm{d}\lambda_b$ is strictly increasing in b.*

Note that by the ergodic theorem $\int c_b \, \mathrm{d}\lambda_b$ is the long-run average of individual consumption. Lemma 2 is very intuitive: the average consumption of an agent increases if the agent receives a larger transfer every day. However, the lemma is not obvious since the agent's strategy maximizes discounted utility instead of average consumption. Standard arguments for proving comparative statics on the invariant distribution of a Markov transition with respect to a change in a parameter require that the transition be monotone in the parameter. Such changes are called positive shocks in Acemoglu and Jensen [1] and underline all the comparative-statics results in their paper and, to our knowledge, all other known similar results. In our setup, however, increasing b is not a positive shock—the problem is that $\chi_b(w)$ is not increasing in b in sense of first-order stochastic dominance.

Here is a rough intuition for the proof: Consider two agents, one facing a transfer b and another b' with $b < b'$. We call the former agent Low and the latter High. We couple the stochastic processes of consumption and saving of High and Low by assuming they receive the same daily endowment. Consider a typical realization of this process. Each day on which High starts with higher wealth than Low, it follows from properties of the optimal consumption that High consumes more than that of Low. Consider now a maximal sequence of consecutive days at the beginning of which High's wealth is lower than Low. Then during these days High paid less taxes than Low, and received the same daily endowment and higher transfer. Moreover, High entered this sequence with more wealth than Low and finished it with less wealth. It then follows that in all these days taken together High consumed more than Low. Therefore, in the long-run High consumes more than Low on every realization. By the ergodic theorem, the long run average consumption on each realization equals the expected daily consumption of the agents. Therefore, the expected consumption of High is higher than that of Low. This is the basic idea of the proof with two caveats: First, we need to be more careful with what happens in the endpoints of the sequence of days in which High has lower wealth than Low. Second, the ergodic theorem delivers the right intuition but it is overkill. In fact, we only need the more basic Kac's Lemma.

Remark 2. It is instructive to consider what happens when we apply the coupling argument to the case of positive shock to the parameter, as in Acemoglu and Jensen [1]. Here again we would have two agents, Low and High, but, under a positive shock, the coupling will deliver a process according to which High's state is always higher than Low's state. Using the ergodic theorem again we get that the stationary distribution of High's state first-order stochastically dominates the stationary distribution of Low's state. This is in contrast to our case, where in the coupled process High may have a lower state than Low on some days.

Proof of Lemma 2. Let $b < b'$, and consider a coupling,

$$(W_0, W_0', Y_0), (W_1, W_1', Y_1), \ldots, (W_t, W_t', Y_t), \ldots,$$

of the wealth process of the two agents with the same endowment process Y_t, where one agent, called Low, receives transfer b at every period, and a second, called High, receives transfer b'. The wealth of the agents at the beginning of period t is W_t and W_t'. Therefore $W_t = \eta_b(W_{t-1}, Y_{t-1})$ and $W_t' = \eta_{b'}(W_{t-1}', Y_{t-1})$. By the same argument as in Claim 2 the process is ergodic.[7] We have to prove that $\mathbb{E}\{c_b(W_0)\} > \mathbb{E}\{c_{b'}(W_0')\}$. By Lemma 1, we

[7] One has to apply the argument in Claim 2 to show that the probability transition over $[\underline{w}_b, \overline{w}_b] \times [\underline{w}_{b'}, \overline{w}_{b'}]$ is uniquely ergodic.

need to prove that

$$\mathbb{E}\left(1_{\{W_0<W_0'\}}\sum_{t=0}^{T-1}(c_{b'}(W_t')-c_b(W_t))\right)>0,$$

where $T=\inf\{1\le t<\infty : W_t<W_t'\}$. Note that the event $W_0<W_0'$ is indeed of positive probability, which follows from the assertion about the support of λ_b and $\lambda_{b'}$ in Claim 2 and the fact that $\underline{w}_b<\underline{w}_{b'}$. We prove the stronger assertion that

$$\sum_{t=0}^{T-1}(c_{b'}(W_t')-c_b(W_t))>0 \text{ a.s.} \tag{9}$$

on the event $\{W_0<W_0'\}$. Indeed, if $T=1$, then the inequality follows from the fact that $W_0<W_0'$ and the monotonicity of $c_b(w)$ (both in w and in b). Assume now that $T>1$. Let $0<t<T$. Then $W_t\ge W_t'$ and therefore

$$s_b(W_t)>s_{b'}(W_t') \tag{10}$$

from the monotonicity of $s_b(w)$ (increasing in w and decreasing in b). In addition, (10) holds for $t=0$ since $W_1=(1-\tau)s_b(W_0)+b$ and $W_1'=(1-\tau)s_{b'}(W_0')+b'$, but $b<b'$ and $W_1\ge W_1'$ since $T>1$. Thus,

$$\begin{aligned}\sum_{t=0}^{T-1}c_b(W_t) &= W_0-\tau\sum_{t=0}^{T-2}s_b(W_t)+\sum_{t=0}^{T-2}Y_t+(T-1)b-s_b(W_{T-1})\\ &< W_0'-\tau\sum_{t=0}^{T-2}s_{b'}(W_t')+\sum_{t=0}^{T-2}Y_t+(T-1)b'-s_{b'}(W_{T-1}')=\sum_{t=0}^{T-1}c_{b'}(W_t'),\end{aligned}$$

where the equalities follow from the aggregation of the households' consumption and transfers from the beginning of period 0 until the market on day $T-1$ closes, and the inequality follows from (10) and the fact that $W_0<W_0', b<b'$. This proves (9). $\square$

Remark 3. The same proof can be used to prove a more general property in our setting: the long-run average consumption in increases when the endowment increases in first-order stochastic dominance. The only change we need to make in the proof is to couple endowment shocks of the two agents so that the high agent receives a higher endowment every period.

3.4 Proof of Theorem 1

The following claims state the implications of condition (2) in Theorem 1 in terms of the individual agent's problem: Claim 3 implies that, when the lump-sum transfer is zero, the agent will save some money after receiving a high endowment, and Claim 4 implies that in this case the agent will save money under the invariant distribution.

Claim 3. *The condition* (2) *holds if and only if* $s_0(\overline{y}) > 0$.

Proof. If (2) holds, then it follows from Euler's equation that the optimal consumption c_0 satisfies $c_0(\overline{y}) < \overline{y}$ so that $s_0(\overline{y}) = \overline{y} - c_0(\overline{y}) > 0$. If (2) does not hold, then the function $\tilde{c}_0(w) = w$ for every $w \in [\underline{y}, \overline{y}]$ satisfies Euler's equation and the transversality condition and therefore it is optimal. Therefore, the optimal s_0 satisfies $s_0(\overline{y}) = \overline{y} - c_0(\overline{y}) = 0$ □

Claim 4. *Condition* (2) *holds if and only if*

$$\int s_0 \; \mathrm{d}\lambda_0 > 0. \tag{11}$$

Proof. Note that $\int s_0 \; \mathrm{d}\lambda_0 > 0$ if and only if $s_0(\overline{w}_0) > 0$, because $\overline{w}_0$ is the maximal element in support(λ_0) and $s_0(\cdot)$ is monotone and continuous.

We use Claim 3. If (2) holds, then $s_0(\overline{y}) > 0$, and therefore $\eta_0(\overline{y}, \overline{y}) > \overline{y}$ which implies $\overline{w}_0 > \overline{y}$ by the definition of $\overline{w}_0$ and Claim 1. Therefore, $s_0(\overline{w}_0) \geq s_0(\overline{y}) > 0$ by the monotonicity of $s_0(w)$. If (2) does not hold, then $s_0(\overline{y}) = 0$, and therefore $\eta_0(\overline{y}, \overline{y}) = \overline{y}$, which implies $\overline{w}_0 = \overline{y}$ by definition of $\overline{w}_0$. Therefore, in this case $s_0(\overline{w}_0) = s_0(\overline{y}) = 0$. □

We are now ready to complete the proof of Theorem 1. It follows from the argument in Section 3.3 that λ_b is the unique distribution that satisfies Condition 1 (invariance) in Definition 3 for every $b \geq 0$.

We claim that there exists a unique $b \geq 0$ such that Condition 2 (government balance) is satisfied. We consider two cases. First, suppose that $\tau = 0$. Then, clearly $b = 0$ is such a unique b. Suppose then that $\tau > 0$. From the invariance of λ_b, we get that

$$\int w \ \lambda_b(\mathrm{d}w) = (1-\tau) \cdot \int s_b \ \mathrm{d}\lambda_b + b + \int y \ \pi(\mathrm{d}y).$$

Since $\int w \ \lambda_b(\mathrm{d}w) = \int s_b \ \mathrm{d}\lambda_b + \int c_b \ \mathrm{d}\lambda_b$ (which follows from $w = c_b(w) + s_b(w)$), we get that

$$b - \tau \cdot \int s_b \ \mathrm{d}\lambda_b = \int c_b \ \mathrm{d}\lambda_b - \int y \ \pi(\mathrm{d}y)$$

for every $b \geq 0$. The assertion now follows from Lemma 2, and the fact that $\int c_b \ \mathrm{d}\lambda_b \xrightarrow[b \to \infty]{} \infty$, since $c_b(\underline{w}_b) \geq \underline{w}_b \geq b$ for every $w \geq b$.

Finally, we need to prove that Condition 3 in Definition 3 (positive saving) holds if and only if (2) holds. Recall that by Claim 4, condition (2) is equivalent to (11). Again we consider two cases. If $\tau = 0$, then $b = 0$ and so a positive saving holds iff (11) holds. Suppose now that $\tau > 0$. To satisfy the government budget balancedness, both (11) and a positive saving are equivalent to $b > 0$, and the result follows immediately from Claim 4. This completes the proof of Theorem 1. Note that we only need the machinery of Lemma 2 when $\tau > 0$.

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