Decentralizing Constrained-Efficient Allocations in the Lagos-Wright Pure Currency Economy*

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Abstract

This paper offers two ways to decentralize the constrained-efficient allocation of the Lagos-Wright (2005) pure currency economy. The first way has divisible money, take-it-or-leave-it offers by buyers, and a transfer scheme financed by money creation. If agents are sufficiently patient, the first best is achieved for finite money growth rates. If agents are impatient, the equilibrium allocation approaches the constrained-efficient allocation asymptotically as the money growth rate tends to infinity. The second way has indivisible money, take-it-or-leave-it offers by buyers, and no government intervention. We discuss the strict implementation of constrained-efficient allocations and the applicability of our scheme to economies with Lucas trees, endogenous participation, match-specific heterogeneity, and sequential competitive markets.

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1 Introduction

The pure currency economy of Lagos and Wright (2005) is explicit about the frictions – lack of commitment/enforcement and absence of public record-keeping – that rule out credit arrangements, making it amenable to normative analysis and mechanism design. One can characterize the set of allocations that are incentive feasible given the frictions; one can identify the allocations that maximize social welfare in that set; and one can look for ways to decentralize the constrained-efficient allocations.

Hu, Kennan, and Wallace (2009) were the first to characterize the set of incentive-feasible allocations in the Lagos-Wright environment. They provided a mechanism, based on planner's proposals, to implement such allocations by assigning the whole match surplus to the consumer if his real balances are equal or greater than a socially-desirable level, and to the producer otherwise. This trading mechanism does not correspond to any bargaining solution (such as Nash or proportional bargaining), and it is not the outcome of standard extensive-form bargaining games.[1]

In this paper we describe two alternative ways to decentralize the constrained-efficient allocation in the Lagos-Wright environment. First, we show that when money is perfectly divisible the constrained-efficient allocation can be decentralized with an extensive form bargaining game whereby buyers make take-it-or-leave-it offers to sellers and an incentive-compatible transfer scheme financed with money creation. This scheme is regressive and specifies that agents with money holdings above a threshold receive a flat transfer.[2] If agents are sufficiently patient, the first best is achieved for positive but finite money growth rates. If agents are impatient, the equilibrium allocation approaches the constrained-efficient allocation asymptotically as the money growth rate tends to infinity.

Second, constrained-efficient allocations can be implemented with indivisible money, a constant money supply, and the same take-it-or-leave-it bargaining game described above. This second scheme does not require any government intervention in the form of transfers financed with money creation. Moreover, it implements uniquely allocations that are arbitrarily close to the constrained-efficient ones.

Indivisible money can decentralize the constrained-efficient allocation with a simple bargaining game and a constant money supply for the following reason. When agents choose their money holdings they take into account its holding cost as measured by the difference between their rate of time preference and the rate of return of money. However, because this holding cost is not a welfare loss for society — money has to be

[1] Gu and Wright (2016) gives an alternative trading mechanism than the one in Hu, Kennan, and Wallace (2009) to implement the constrained efficient allocation. This alternative mechanism does not correspond to any known bargaining solution either, but it satisfies a monotonicity axiom that is not satisfied by the Hu-Kennan-Wallace mechanism.

[2] This scheme is related, but different, from the affine transfer schemes proposed by Andolfatto (2010) and Wallace (2014) for competitive economies.

held by someone — agents choose inefficiently low real balances. By making money indivisible, the social planner prevents agents from reducing their money holdings below the socially optimal level and reduces their problem to a binary choice between the constrained-efficient allocation and autarky.

Finally, we discuss in conclusion the applicability of our decentralization scheme to economies with Lucas trees, endogenous participation, match-specific heterogeneity, and sequential competitive markets.

2 The model

The basic setup is borrowed from Lagos and Rocheteau (2005) and Rocheteau and Wright (2005). Time is discrete, goes on forever, and is indexed by $t \in \mathbb{N}_0$. The economy is populated by a $[0, 2]$ continuum of infinitely-lived agents, divided evenly into a set of *buyers,* denoted $\mathbb{B}$, and a set of *sellers,* denoted $\mathbb{S}$. Each date has two stages: the first stage has a decentralized market (DM) where agents are matched bilaterally and at random, and the second stage has centralized meetings (CM).[3] In each DM, the measure of pairwise meetings is $\sigma \in [0, 1]$.

There is a single perishable good produced in each stage, with the CM good taken as the numéraire. In the CM, all agents have the ability to produce and wish to consume. Agents' labels as buyer or seller correspond to their roles in the DM where only sellers are able to produce and only buyers wish to consume. Buyers' preferences are represented by the following utility function

$$\mathbb{E} \sum_{t=0}^{\infty} \beta^t \left[u(q_t) + x_t - h_t \right],$$

where $\beta \equiv (1 + r)^{-1} \in (0, 1)$ is the discount factor, q_t is DM consumption, x_t is CM consumption, and h_t is the supply of hours in the CM. Sellers' preferences are given by

$$\mathbb{E} \sum_{t=0}^{\infty} \beta^t \left[-c(q_t) + x_t - h_t \right].$$

The buyer's utility of DM consumption, $u(q)$, is increasing and concave, and the seller's disutility of production, $c(q)$, is increasing and convex, with $u(0) = c(0) = 0$. The surplus function, $u(q) - c(q)$, is strictly concave, with $q^* = \arg\max \left[u(q) - c(q) \right]$. Moreover, $u'(0) = c'(\infty) = \infty$ and $c'(0) = u'(\infty) = 0$. All agents have access to a linear technology to produce the CM output from their own labor, $x = h$.

There is also an intrinsically useless, perfectly durable asset called money. Money can be either divisible or indivisible. Lack of record-keeping and private information about individual trading histories rule out unsecured credit, giving a role for money to serve as means of payment. In addition, individual asset holdings

[3] One can think of agents as being ex-ante identical and their type as buyer or seller being realized with equal probabilities at the beginning of the CM. In contrast, in Lagos and Wright (2005) the type is realized in the DM when pairwise meetings are formed. While these two formulations generate differences for implementation (see, e.g., Rocheteau, 2012), our main proposition would be unaffected.

are common knowledge in a match. We assume that sellers do not carry real balances across periods. It is with no loss in generality for our normative analysis, and it would be easy to check that sellers have no incentive to acquire money in the equilibria we will characterize.

We study equilibrium outcomes that can be implemented with a *planner's proposal*. Under divisible money, a *proposal* consists of four objects: *(i)* an initial distribution of money, μ, and a gross money growth rate, γ; *(ii)* a sequence of transfer policies, $\tau_t : \mathbb{R}_+ \to \mathbb{R}_+$, where each $\tau_t(m)$ specifies a transfer of money (expressed in the numéraire) from the government to buyers at the start of period t (before DM trades);[4] *(iii)* a sequence of CM prices for money, $\{\phi_t\}_{t=1}^{\infty}$, in terms of the numéraire; *(iv)* a sequence of functions in the bilateral matches, $o_t : \mathbb{R}_+ \to \mathbb{R}_+^2$, each of which maps the buyer's money holdings, m_t, into a proposed trade, $(q_t, d_t) \in \mathbb{R}_+ \times [0, \phi_t \times m_t]$, where q_t is the DM output produced by the seller and consumed by the buyer and d_t is the transfer of money from the buyer to the seller (expressed in terms of the numéraire). As in Wallace (2014), $\tau_t(m) \geq 0$ because there is no enforcement in pure currency economies and $\tau_t(m)$ is weakly increasing for the scheme to be incentive-compatible.

Under indivisible money, a *proposal* is a similar object but we assume money supply is constant. Moreover, $o_t : \mathbb{R}_+ \to \mathbb{R}_+^2$, maps the buyer's money holdings, m, into $(q_t, \ell_t) \in \mathbb{R}_+ \times \Delta(\{0, ..., m_t\})$, where ℓ_t is a lottery over monetary transfers, with $\ell_t(m)$ denoting the probability of transferring m units of money, and $d(\ell_t)$ denoting its expected value in terms of the numéraire good.

The trading procedure in the DM is given by the following game. Given the buyer's money holdings and the associated proposed trade, $o_t(m_t)$, both the buyer and the seller simultaneously respond with *yes* or *no*: if both respond with *yes*, then the proposed trade is carried out; otherwise, there is no trade. Since both agents can turn down the proposed trade, this ensures that trades are individually rational. We also require any proposed trade to be in the pairwise core.[5] Agents in the CM trade competitively against the proposed prices, which is consistent with the pairwise core requirement in the DM due to the equivalence between the core for the centralized meeting and competitive equilibria.

The strategy of a buyer, s_b, consists of two components for any given trading history h^t at the beginning of period t: *(i)* $s_b^{h^t,1}(m) \in \{yes, no\}$ that maps the buyer's money holdings in a DM meeting, m, to his response to the offer o_t; *(ii)* for the divisible money case, $s_b^{h^t,2}(m, a_b, a_s) \in \mathbb{R}_+$ maps the buyer's money holdings, m, the buyer's and seller's choices of whether to accept or reject the trade, $a_b, a_s \in \{yes, no\}$, to

[4] Here we restrict the policy to make transfers after the CM trades only to buyers, but we can also introduce transfers after agents' DM trades or transfers to sellers. This addition does not change the necessary condition for implementation and we show in our results that we can implement the constrained efficient allocation with our policy.

[5] The pairwise core requirement can be implemented directly with a trading mechanism that adds a renegotiation stage as in Hu, Kennan, and Wallace (2009), following the *yes* responses from both agents. The renegotiation stage will work as follows. An agent will be chosen at random to make an alternative offer to the one made by the mechanism. The other agent will then have the opportunity to choose between the two offers.

his final money holdings in the CM; for the indivisible money case, $s_b^{h^t,2}(m, a_b, a_s) \in \Delta(\mathbb{N}_0)$ maps m and (a_b, a_s) to a distribution of his final money holdings in the CM. The strategy of a seller at the beginning of period-t is a function, $s_s^{h^t}(m) \in \{yes, no\}$, that represents the seller's response to the offer $o_t(m)$.

Remark 1 *Under indivisible money, we could allow the buyer to randomize his money holdings in the CM, or introduce separate markets for lotteries. Under no arbitrage condition, these two are equivalent as the price for a lottery $\ell \in \Delta(\mathbb{N})$ over money transfers in the CM is $\sum_{m=0}^{\infty} \ell(m) m\phi$.*

A *simple equilibrium* is a list, $\langle (s_b : b \in \mathbb{B}), (s_s : s \in \mathbb{S}), \mu, \{o_t, \phi_t, \tau_t\}_{t=1}^{\infty} \rangle$, composed of one strategy for each agent and the proposal $(\mu, \{o_t, \phi_t, \tau_t\}_{t=1}^{\infty})$ such that: (i) each strategy is sequentially rational given other players' strategies; and (ii) the centralized market clears at every date. We require the real balances to be stationary in the sense that $\phi_t = \phi_{t+1}\gamma$ and the transfers are financed through inflation:

$$\int \tau_t(m_t) \mathrm{d}F_t(m_t) = (\gamma - 1)\, \phi_t M_t, \tag{1}$$

where F_t is the equilibrium distribution of money holdings across buyers at the beginning of date t, and M_t is the average money holding (per buyer) at the beginning of date t (both before the money transfers). In what follows, we focus on stationary planner proposals where real balances are constant over time and equilibria such that *(i)* agents follow symmetric and stationary strategies; *(ii)* agents always respond with *yes* in all DM meetings; and *(iii)* the initial distribution of money is uniform across buyers. In the divisible money case, we also require the proposals τ_t and o_t to be stationary in the sense that they depend on the nominal money holdings only through real balances and are constant when treated as a function of real balances. Thus, in a simple equilibrium, the proposed DM trades, $o_t(m_t)$ can be written as $o(z) = [q(z), d(z)]$ with $z = m_t\phi_t$, the real balance, and $\tau_t(m_t)$ can be written as $\tau(z)$ (note that for τ, its output is also measured in terms of real balances).

The outcome of a simple equilibrium is summarized by a list, (q^p, d^p, z^p), where (q^p, d^p) are the terms of trade in the DM and z^p is the buyer's real balances when exiting the CM.[6] An outcome (q^p, d^p, z^p) is said to be *implementable* if it is the equilibrium outcome of a simple equilibrium associated with proposed DM trades o and a transfer policy τ under divisible money or associated with proposed DM trades o under indivisible money, and if the terms of trade satisfies the pairwise core requirement. The ex-ante social welfare of an outcome (q^p, d^p, z^p) is given by

$$\mathcal{W}(q^p, d^p, z^p) = \frac{\sigma[u(q^p) - c(q^p)].}{1 - \beta}$$

[6] Although we allow mixed strategies in the portfolio choice in the CM, we only consider equilibrium outcomes in which all buyers leave the CM with the same amount of money. Our claim about constrained efficiency is not affected if we also consider equilibrium outcomes in which buyers leave the CM with the same distribution of money holdings.

An outcome is said to be *constrained efficient* if it maximizes the welfare subject to implementability.

Under divisible money and a given mechanism, the value function of a buyer holding z real balances at the beginning of the CM satisfies

$$W^b(z) = z + \max_{z' \geq 0} \left\{ -\gamma z' + \beta V^b \left[z' + \tau(z') \right] \right\}, \tag{2}$$

where $V^b(z)$ is the value function of a buyer holding z real balances in the DM after the government transfers, τ, have been implemented. To obtain (2) we used the budget constraint according to which next-period real balances are $z' = \gamma^{-1}(h - x + z)$, so that the post-transfer real balances are $z' + \tau(z')$, where γ^{-1} is the gross rate of return of money. Under indivisible money and a given mechanism, the value function of a buyer holding z real balances at the beginning of the CM satisfies

$$W^b(z) = z + \max_{\ell' \in \Delta(\{0,1,2,\ldots\})} \sum_{m=0}^{\infty} \ell'(m) \left\{ -m\phi + \beta V^b(m\phi) \right\}, \tag{3}$$

where ϕ is the CM price for money.[7] The difference between (2) and (3) is that in the latter money holdings are restricted to $\mathbb{N}_0$ but agents are allowed to randomize over $\mathbb{N}_0$.

The Bellman equation in the DM is

$$V^b(z) = \sigma \left\{ u\left[q(z) \right] + W^b[z - d(z)] \right\} + (1 - \sigma) W^b(z). \tag{4}$$

The buyer meets a producer with probability σ, in which case he consumes $q(z)$ and makes a payment $d(z)$ expressed in terms of the numéraire. When money is indivisible this payment corresponds to the expected value of a lottery, ℓ. There is a similar set of value functions for sellers, i.e.,

$$\begin{aligned} W^s(z) &= z + \beta V^s, \\ V^s &= \sigma \left[-c(q^p) + W^s(d^p) \right] + (1 - \sigma) W^s(0), \end{aligned}$$

since, with no loss in generality, sellers do not carry money from the CM to the DM, and on the equilibrium path all DM trades are given by the outcome (q^p, d^p).

3 Main results: Implementation

The following proposition characterizes the set of incentive-feasible output levels for the environments with divisible and indivisible money. For each environment we propose a way to decentralize the constrained-efficient allocation with take-it-or-leave-it offers by buyers.

[7] Alternatively, we could adopt use the formulation with sunspot states and a complete set of state-contigent commodities, as in Rocheteau, Rupert, Shell and Wright (2008). Our remark 1 is equivalent to the statement that every sunspot equilibrium allocation can be supported by prices, when adjusted for probabilities, that are constant across states. In this case $\ell(m)$ is interpreted as the measure of states where agents choose to purchase m units of money.

Proposition 1 ***(Implementation of Constrained-Efficient Allocations.)*** *The constrained-efficient output level in pairwise meetings under both divisible and indivisible money is*

$$q^c \equiv \max\left\{q : -rc(q) + \sigma[u(q) - c(q)] \geq 0\right\}. \tag{5}$$

(a) Divisible money. *If $r < \sigma\left[u(q^*) - c(q^*)\right]/c(q^*)$, then there is a $\underline{\gamma} \in (1, +\infty)$ and a transfer scheme financed with money creation,*

$$\tau(z) = \begin{cases} 0 & \text{if } z < c(q^p)/\gamma \\ (1-\gamma^{-1})c(q^p) & \text{otherwise,} \end{cases} \tag{6}$$

such that for all $\gamma \geq \underline{\gamma}$, $q^p = q^*$ is decentralized with buyer's take-it-or-leave-it offers.

If $r \geq \sigma\left[u(q^*) - c(q^*)\right]/c(q^*)$, then for all $\gamma > 1$, there is a $q^p_\gamma < q^*$ decentralized with buyer's take-it-or-leave-it offers and a policy analogous to (6). Moreover, q^p_γ is increasing with γ and $\lim_{\gamma\to+\infty} q^p_\gamma = q^c$.

(b) Indivisible money. *The constrained-efficient allocation is decentralized with indivisible money, $M = 1$, and buyer's take-it-or-leave-it offers.*

Proof. The necessary condition for implementability, $-rc(q)+\sigma[u(q)-c(q)] \geq 0$, follows from Rocheteau (2012). The argument applies to both the economy with divisible money and the economy with indivisible money, and we only give a sketch for the divisible case here. Suppose that (q^p, d^p, z^p) is an implementable outcome. The seller's participation constraint in the DM is $c(q^p) \leq d^p$. A necessary condition for buyer's participation in the CM is that he prefers going along with the planner's proposal by accumulating z^p rather than leaving the CM with zero real balances and saying "no" to the following DM trade (two consecutive deviations), i.e.,

$$-z^p + \beta\left\{\sigma\left[u(q^p) - d^p\right] + \frac{z^p}{\gamma} + \tau\left(\frac{z^p}{\gamma}\right)\right\} \geq \beta\tau(0),$$

where $\tau(0)$ is the transfer of real balances received by the buyer who deviates by holding no real balances. Using the budget constraint, (1), $\tau(z^p/\gamma) = (1-\gamma^{-1})z^p$, the inequality above can be rewritten as:

$$-rz^p + \sigma\left[u(q^p) - d^p\right] \geq \tau(0).$$

Together with the assumption of no enforcement, $\tau(0) \geq 0$, the government budget constraint, $\tau(z^p/\gamma) = (1 - 1/\gamma)\, z^p$, and $c(q^p) \leq d^p \leq z^p/\gamma + \tau\left(z^p/\gamma\right) = z^p$, it implies the inequality in (5). The highest $q \in [0, q^*]$ solution to $-rc(q)+\sigma[u(q)-c(q)] \geq 0$ can be implemented with planner's offers as described in Hu, Kennan, and Wallace (2009).

(a) Decentralization with divisible money. We now show that under divisible money the constrained-efficient allocation is implemented with take-it-or-leave-it offers by buyers, $d(z) = c[q(z)] = \min\{z, c(q^*)\}$, and the transfer scheme (6). From (4), and using the linearity of W^b, the value of the buyer in the DM solves

$$V^b(z) = \sigma \{u[q(z)] - c[q(z)]\} + W^b(z). \tag{7}$$

Substituting $V^b(z)$ into (2) the buyer's choice of real balances (assuming a solution exists) is given by $\max\left\{\tilde{U}(\gamma), \hat{U}(\gamma)\right\}$ where

$$\begin{aligned}
\tilde{U}(\gamma) &= \sup_{z<\gamma^{-1}c(q^*)} \left\{-\left(\frac{\gamma-\beta}{\beta}\right)z + \sigma\left[u \circ c^{-1}(z) - z\right]\right\} \\
\hat{U}(\gamma) &= \max_{z\geq\gamma^{-1}c(q^*)} \left\{-\left(\frac{\gamma-\beta}{\beta}\right)z + \sigma\left[u(q^*) - c(q^*)\right] + (1-\gamma^{-1})c(q^*)\right\},
\end{aligned}$$

where $\tilde{U}(\gamma)$ is the net expected utility of the buyer if he does not receive the transfer and $\hat{U}(\gamma)$ is his net utility if he receives the transfer, which requires that he holds at least $\gamma^{-1}c(q^*)$ real balances. It is clear that

$$\hat{U}(\gamma) = \hat{U} \equiv -rc(q^*) + \sigma\left[u(q^*) - c(q^*)\right].$$

Note that $\tilde{U}(\gamma)$ is strictly decreasing in γ, $\tilde{U}(1) > \hat{U}$, and it converges to zero as γ tends infinity. Hence, provided that $-r + \sigma\left[u(q^*) - c(q^*)\right]/c(q^*) > 0$, so that $\hat{U} > 0$, there is a $\underline{\gamma} \in (1, +\infty)$ such that for all $\gamma \geq \underline{\gamma}$, $\tilde{U}(\gamma) \leq \hat{U}$. Buyers accumulate $z = \gamma^{-1}c(q^*)$ real balances, which become $c(q^*)$ after the policy, and trade the first best in the DM.

Consider next the case $-r + \sigma\left[u(q^*) - c(q^*)\right]/c(q^*) \leq 0$. For any given $\gamma > 1$ denote q^p_γ the largest q that solves:

$$-rc(q^p_\gamma) + \sigma\left[u\left(q^p_\gamma\right) - c\left(q^p_\gamma\right)\right] = \max_{z\geq 0}\left\{-\left(\frac{\gamma-\beta}{\beta}\right)z + \sigma\left[u \circ c^{-1}(z) - z\right]\right\}. \tag{8}$$

Suppose $\tau(z) = (1-\gamma^{-1})c(q^p_\gamma)$ if $z \geq \gamma^{-1}c(q^p_\gamma)$ and $\tau(z) = 0$ otherwise. The buyer's net utility from choosing $z \geq \gamma^{-1}c(q^p_\gamma)$ is

$$\hat{U}(\gamma) = \max_{z\geq\gamma^{-1}c(q^p_\gamma)}\left\{-\left(\frac{\gamma-\beta}{\beta}\right)z + \sigma\{u[q(\tilde{z})] - c[q(\tilde{z})]\} + (1-\gamma^{-1})c(q^p_\gamma)\right\},$$

where $\tilde{z} = z + (1-\gamma^{-1})c(q^p_\gamma)$. By the definition of q^p_γ it follows that $-\left(\frac{\gamma-\beta}{\beta}\right)z + \sigma\{u[q(\tilde{z})] - c[q(\tilde{z})]\}$ is decreasing in z for all $z \geq \gamma^{-1}c(q^p_\gamma)$. Hence the optimal choice is $z = \gamma^{-1}c(q^p_\gamma)$ and

$$\hat{U}(\gamma) = -rc(q^p_\gamma) + \sigma\left[u\left(q^p_\gamma\right) - c\left(q^p_\gamma\right)\right].$$

By the same reasoning as above, for all $\gamma' \geq \gamma$ buyer's optimal choice of real balances is $z = \gamma^{-1}c(q^p_\gamma)$. As γ tends to infinity, the right side of (8) approaches 0 and q^p_γ approaches q^c by below.

(b) Decentralization with indivisible money. Now we decentralize the constrained-efficient allocation with indivisible money. The proposal has all buyers receive exactly one unit of money, $M = 1$ at $t = 0$, the price for money in CM is $\phi = c(q^c) \leq c(q^*)$, and in each DM meeting, the buyer who holds m units of money makes a take-it-or-leave-it offer (without lotteries) to the seller. The buyer's problem is:

$$\max_{q,\ell \in \Delta(\{0,\ldots,m\})} \left[u(q) - \sum_{m'=0}^{m} \phi \ell(m')m' \right] \quad \text{s.t.} \quad -c(q) + \sum_{m'=0}^{m} \phi \ell(m')m' = 0. \tag{9}$$

Note that because of quasi-linearity, we may replace the lottery with its expected value d (in terms of the numéraire) which ranges (continuously) from 0 to ϕm. The value function for a buyer holding m units of money is $V^b(\phi m)$ that solves (7). Substituting $V^b(\phi m)$ into the buyer's CM value function, his CM problem can be re-expressed as

$$\max_{m \in \mathbb{N}_0} \left\{ -r\phi m + \sigma \max_{d \in [0,\phi m]} \left[u \circ c^{-1}(d) - d \right] \right\}, \tag{10}$$

where $r = (1-\beta)/\beta$.

Since $u \circ c^{-1}(\cdot)$ is strictly concave in m, the buyer's maximization problem (10) would have a unique solution *if money were perfectly divisible*. This solution is denoted as $m^* \in \mathbb{R}_+$. However, because money is indivisible, this solution may not be feasible. Let $[m^*]$ denote the integer part of m^*. Consequently, (10) has, at most, two solutions which are $[m^*]$ and $[m^*] + 1$.

The market clearing condition in the CM requires that buyers prefer holding one unit of money instead of two or zero. The condition for preferring one unit of money to two is

$$-r\phi + \sigma \left[u \circ c^{-1}(\phi) - \phi \right] \geq -r2\phi + \sigma \left[u(q_2) - c(q_2) \right], \tag{11}$$

where $q_2 = \min\{q^*, c^{-1}(2\phi)\}$. The condition for preferring one unit of money to zero is given by

$$-r\phi + \sigma \left[u \circ c^{-1}(\phi) - \phi \right] \geq 0. \tag{12}$$

It then suffices to show that $\phi = c(q^p) \leq c(q^*)$ satisfies (11) and (12).

We separate two cases.

(a) $q^p = q^*$. Condition (11) collapses to $-rc(q^*) \geq -r2c(q^*)$, which is satisfied. Condition (12) can be rewritten as

$$c(q^*) \leq \frac{\sigma}{r+\sigma} u(q^*), \tag{13}$$

and this follows directly from (5).

(b) $q^p < q^*$. In this case, $-r\phi + \sigma[u \circ c^{-1}(\phi) - \phi] = 0$, which implies (12) holds. For any $\phi' > \phi$, $-r\phi' + \sigma[u \circ c^{-1}(\phi') - \phi'] < 0$. Hence, if $2\phi < c(q^*)$, (11) is satisfied since

$$-r2\phi + \sigma \left[u \circ c^{-1}(2\phi) - 2\phi \right] < 0 = -r\phi + \sigma \left[u(q^p) - c(q^p) \right].$$

Now suppose that $2\phi \geq c(q^*) > \phi = c(q^p)$. Since $-r\phi' + \sigma[u \circ c^{-1}(\phi') - \phi'] < 0$ for all $\phi' > \phi$, $-rc(q^*) + \sigma[u(q^*) - c(q^*)] < 0$, and hence (11) is satisfied as $2\phi \geq c(q^*)$. Finally, note that since buyers are willing to hold one unit against zero or two, he is willing to hold one unit against any mixed strategy over money holdings. □

Constrained-efficient allocations with divisible money coincide with constrained-efficient allocations with indivisible money. Those allocations can be implemented by planner's proposals, as in Hu, Kennan and Wallace (2009), that specify that the buyer receives the whole match surplus if he holds $z \geq c(q^c)$; otherwise, the buyer receives no surplus.

Proposition 1 offers two alternative ways to decentralize constrained-efficient allocations with an extensive-form bargaining game whereby the buyer makes take-it-or-leave-it offers to sellers. If money is divisible, buyers can be incentivized to hold the socially-optimal real balances by receiving a transfer of money if $z \geq c(q^c)/\gamma$ and no transfer otherwise.[8] This transfer exactly compensates agents who hold the socially optimal real balances for the inflation tax, while the inflation tax makes deviations costly. If $q^c = q^*$ this transfer scheme can be financed with a finite money growth rate. If $q^c < q^*$, the constrained-efficient allocation is only approached asymptotically as the money growth rate tends to infinity.[9] Intuitively, inflation must be infinite so that a buyer who defects and holds less than $c(q^c)/\gamma$ ends up with no real balances in the next DM and therefore receives no surplus. Output and social welfare are monotone increasing with the money growth rate when implemented through τ. This is in contrast with the negative welfare effects of inflation engineered through lump-sum transfers. However, this scheme works as intended only if groups of agents cannot exploit the regressive transfers by pooling their money holdings while holding less than $c(q^c)$. We rule out such group deviations by assuming that transfers occur at the start of the period when agents cannot meet each other.

If money is indivisible, then the buyer's take-it-or-leave-it bargaining game implements the constrained-efficient allocation without any government intervention: the money supply is constant and there is no transfer. Buyers do not need to be incentivized with a transfer because their only relevant choice is to hold one unit of money or zero. By choosing the appropriate price for money in the CM, the constrained-efficient allocation is incentive feasible and holding one unit of money dominates holding none. Note that although

[8]While it achieves the same purpose, our transfer scheme differs from the one in Andolfatto (2010) in that transfers occur after the CM and before agents are subject to the idiosyncratic DM shocks. (The DM in Andolfatto's model is competitive.) As a result, at the time of the transfer the distribution of money is degenerate, which allows us to use a simple step function and to eliminate the redemption fee. Our scheme is also a limiting version of the regressive scheme in Wallace (2014) where the slope of the transfer is infinite.

[9]This result is in contrast with Andolfatto's (2010) claim that under an appropriately designed policy the equilibrium allocation can be made arbitrarily close to the first-best. The equilibrium allocation can be made arbitrarily close to the constrained-efficient allocation, as defined in Hu, Kennan, and Wallace (2009), which only coincides with the first best when the rate of time preference is sufficiently low.

we allow randomization in the CM, this logic is not affected.

This result illustrates a socially beneficial role of indivisible money. While divisible money expands the set of feasible trades in pairwise meetings (see, e.g., Berentsen and Rocheteau, 2002), it also expands the set of admissible deviations in the CM, allowing for deviations that are individually optimal but not socially optimal. Indeed, buyers choose their real balances to maximize their expected DM surplus net of the cost of holding real balances. Under divisible money and take-it-or-leave-it offers q solves $rc'(q) = \sigma[u'(q) - c'(q)]$. The rate of time preference does not enter the planner's objective, which consists of the sum of the DM match surpluses. As a result, buyers choose lower real balances than the ones that implement the constrained-efficient allocation. Indivisible money makes such deviations unavailable.

If money is indivisible and $M = 1$ there is a range of values for ϕ that are consistent with market clearing. We now show how to implement uniquely a "near-constrained-efficient" allocation by setting $M < 1$, i.e., money is scarce in the sense that not all buyers can hold one indivisible unit of money.[10] Market clearing in the CM implies that buyers must be indifferent between holding one unit of money or zero, i.e.,

$$-r\phi + \sigma\left[u(q) - c(q)\right] = 0,$$

where $q = \min\{c^{-1}(\phi), q^*\}$ by the take-it-or-leave-it offer bargaining game. If $r \geq \sigma\left[u(q^*) - c(q^*)\right]/c(q^*)$, then the value of money coincides with the one at the constrained-efficient allocation, $\phi = c(q^c) \leq c(q^*)$, and lotteries are not used. If $r < \sigma\left[u(q^*) - c(q^*)\right]/c(q^*)$, then $\phi > c(q^*)$. In this case the DM trade is given by (q^*, ℓ) with $\ell(1) = c(q^*)/\phi < 1$.[11] Moreover, the equilibrium price of money is independent of M, i.e., money is neutral. Hence, as M approaches 1 the unique steady-state equilibrium converges to the equilibrium that implements the constrained-efficient allocation.[12]

Proposition 2 ***(Strict Implementation)*** *For all $M < 1$, there is a unique steady-state monetary equilibrium of the economy with indivisible money and take-it-or-leave-it offers by buyers and it is such that $q = q^c$. As $M \nearrow 1$, the equilibrium allocation converges to a constrained-efficient allocation.*

4 Extensions

We conclude with four extensions to discuss the robustness of our decentralization scheme with indivisible money. First, we study an economy with Lucas trees to show that Proposition 1 can be generalized to the

[10]Strict implementation here is in terms of the existence of a unique steady-state equilibrium. As is standard in monetary models, there could be other non-stationary equilibria.

[11]Lotteries are useful when the first-best is implementable with a slack participation constraint, as in Berentsen, Molico, and Wright (2002). If lotteries are not feasible (let say, because agents do not have access to a nonmanipulable randomization device) one can still implement uniquely $q^c = q^*$ by adopting a bargaining solution or a bargaining game that gives sellers some bargaining power.

[12]The equilibrium correspondence is upper hemi continuous at $M = 1$ and the socially desirable allocation can only be obtained as $M \nearrow 1$.

case of real assets (or claims on real assets). Second, we study an economy with endogenous participation as way to illustrate that the buyers-take-all bargaining game does not always decentralize the constrained efficient allocation. Third, we briefly review an economy with match-specific heterogeneity to illustrate conditions under which indivisible money fails to implement constrained-efficient allocations (in the absence of lotteries). Fourth, we consider an economy with sequential competitive markets to provide an example where indivisible money might outperform a transfer scheme.

4.1 Lucas trees

Suppose there is a fixed supply, $A > 0$, of infinitely-lived Lucas trees, where each Lucas tree pays one unit of numéraire in the CM, as in Geromichalos, Licari, and Suarez-Lledo (2007) and Lagos (2010). The constrained-efficient allocation when assets are divisible is characterized in Hu and Rocheteau (2015). Suppose trees are not portable and the planner is the only entity that can produce claims on the trees that can be authenticated in the DM. The planner produces $M = 1$ indivisible claims that yield $\rho = A$ in terms of numéraire each. We decentralize the constrained-efficient allocations with take-it-or-leave-it offers by buyers. We focus on the case where $c(q^*) > (1 + r)\rho/r$ so that the constrained-efficient allocation is not achieved with divisible Lucas trees as means of payment. The buyer's portfolio choice in the CM is:

$$\max_{m \in \mathbb{N}_0} \left\{ -(r\phi - \rho)\, m + \sigma\left[u(q) - c(q)\right] \right\}. \tag{14}$$

The opportunity cost of holding money is reduced by the dividend of the claims on Lucas trees. An equilibrium with $c(q) = c(q^*) = \phi + \rho$ exists if

$$\rho \geq \frac{rc(q^*) - \sigma\left[u(q^*) - c(q^*)\right]}{1 + r}, \tag{15}$$

which is the condition to implement the first best in Hu and Rocheteau (2015). The rate of return of money, $\rho/\phi = \rho/[c(q^*) - \rho]$, is less than r and it is increasing with ρ. If $q = q^*$ is not implementable, then, by the same reasoning, q solves

$$rc(q) = \sigma\left[u(q) - c(q)\right] + (1 + r)\rho,$$

which is the constrained efficient output level.

4.2 Endogenous participation

Suppose now there is endogenous participation of sellers, as in Rocheteau and Wright (2005). Sellers who wish to participate in the DM incur a disutility cost, $k > 0$ and $k < u(q^*) - c(q^*)$. The measure of participating sellers is denoted n and the measure of matches in the DM is $\sigma(n)$, where $\sigma(n)$ is increasing and concave. The planner chooses (q, n) to maximize $\sigma(n)[u(q) - c(q)] - kn$ subject to the participation constraints $-rd + \sigma(n)[u(q) - d] \geq 0$ and $-k + \sigma(n)[d - c(q)]/n = 0$ (assuming $n > 0$). Here we only consider implementation of the first best, under which $n = n^*$ with $\sigma'(n^*)[u(q^*) - c(q^*)] = k$. To do so, we decentralize the first best with the generalized Nash bargaining solution,

$$q = \arg\max_{q \geq 0} [u(q) - \phi]^{\theta} [-c(q) + \phi]^{1-\theta}. \tag{16}$$

where θ is the buyer's bargaining power. The trading mechanism in the previous section, take-it-or-leave-it offers by buyers, corresponds to $\theta = 1$. In order to implement $q = q^*$ the value of the indivisible money is $\phi = \theta c(q^*) + (1-\theta)u(q^*)$, which is consistent with market clearing provided that

$$-r[\theta c(q^*) + (1-\theta)u(q^*)] + \sigma(n^*)\theta[u(q^*) - c(q^*)] \geq 0. \tag{17}$$

The measure of sellers solves the free-entry condition:

$$\frac{\sigma(n)}{n}(1-\theta)[u(q^*) - c(q^*)] = k, \tag{18}$$

where $n = n^*$ if $\theta = 1 - \sigma'(n^*)n^*/\sigma(n^*)$. So generalized Nash bargaining and indivisible money decentralize the first best provided that the Hosios condition holds.

4.3 Match heterogeneity

We introduce match specific heterogeneity as in Lagos and Rocheteau (2005) by assuming that the buyer's utility in the DM is $\varepsilon u(q)$ where $\varepsilon \in [0,1]$ is the realization of a random variable with c.d.f. $F(\varepsilon)$. Assuming that ε is common knowledge in a match, an offer with indivisible money and lotteries is, with a slight abuse of notation, a mapping $o(\varepsilon) = [q(\varepsilon), d(\varepsilon)]$, where $q(\varepsilon)$ is DM output and $d(\varepsilon) \leq \phi$ is the expected real value of the money transfer. The first-best outcome is implementable with buyer's take-it-or-leave-it offers, indivisible money, $M = 1$, and lotteries, if the price, $\phi = c(q_1^*)$, clears the money market:

$$-rc(q_1^*) + \sigma \int [\varepsilon u(q_\varepsilon^*) - c(q_\varepsilon^*)] \, dF(\varepsilon) \geq 0, \tag{19}$$

where $q_\varepsilon^* = \arg\max_q \{\varepsilon u(q) - c(q)\}$ is the first-best output level. For all $\varepsilon \leq 1$, the offer, $o(\varepsilon) = [q_\varepsilon^*, c(q_\varepsilon^*)]$, is feasible with lotteries since $c(q_\varepsilon^*) \leq \phi$. In all matches buyers consume the first best and offer a lottery that compensates sellers for their disutility of production. According to (19) for the market to clear buyers must (weakly) prefer to hold one unit of money instead of 0. This condition requires that the holding cost of money, the first term on the right side of (19), is no greater than the expected surplus in the DM, the second term on the right side of (19). Without lotteries, this outcome is not implementable with indivisible money because this would require $c(q_\varepsilon^*) \leq \phi \leq \varepsilon u(q_\varepsilon^*)$ for all $\varepsilon \in [0,1]$. The value of money must be sufficiently large to cover the seller's disutility in all matches, but it cannot be too large in order to prevent buyers from hoarding their unit of money. Clearly, these inequalities cannot hold for both $\varepsilon = 1$, in which case $\phi \geq c(q_1^*)$, and $\varepsilon \approx 0$, in which case $\phi \leq \varepsilon u(q_\varepsilon^*) \approx 0$.

4.4 Sequential competitive markets

Suppose all markets (CMs and DMs) are competitive, as in Rocheteau and Wright (2005) and Andolfatto (2010). We reinterpret σ as both the probability that a buyer wants to consume in the DM and the probability that a seller is able to produce. Let p denote the price of the DM good in terms of the CM good. Because money is indivisible and we do not allow for lotteries, $q \in \{0, \phi/p, 2\phi/p, ...\}$. Market clearing in the DM requires:

$$\begin{aligned} c\left(\frac{\phi}{p}\right) &\leq \phi \leq u\left(\frac{\phi}{p}\right) &(20)\\ -m\phi + c\left(\frac{m\phi}{p}\right) &\leq -\phi + c\left(\frac{\phi}{p}\right) \text{ for all } m \geq 2. &(21)\end{aligned}$$

Inequalities (20) require that buyers want to sell their unit of money while sellers want to acquire one unit money. Inequality (21) guarantees that sellers do not want to acquire more than one unit of money. With no loss, we restrict our attention to equilibria such that $c(\phi/p) = \phi$, in which case (20)-(21) hold if $q \leq q^*$. Market clearing in the CM requires:

$$\begin{aligned} -r\phi + \sigma\left[u\left(\frac{\phi}{p}\right) - \phi\right] &\geq 0 &(22)\\ -rm\phi + \sigma\left[u\left(\frac{m\phi}{p}\right) - m\phi\right] &\leq 0 \text{ for all } m \geq 2. &(23)\end{aligned}$$

The first best, q^*, is implementable if $c(q^*) \leq \frac{\sigma}{\sigma+r}u(q^*)$. Otherwise, one can implement the highest q such that $c(q) = \frac{\sigma}{\sigma+r}u(q)$. So with indivisible money one can implement the same allocations as the ones of the economy with pairwise meetings. The lack of divisibility of money restricts choices in the CM, thereby preventing agents from choosing inefficiently low real balances. It also prevents deviations by coalitions of agents in the DM, thereby making the core requirement less stringent.

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