Complexity and Mixed Strategy Equilibria: Supplemental Material

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The supplemental material gives three results. The first result shows that, for any oracle $\theta \in \{0,1\}^{\mathbb{N}}$, a θ-incompressible sequence can be used to compute θ-random sequences for any general measure that is independent across periods (Theorem 2.1). The second result shows that for any such θ-random sequence, a version of the Law of Iterated Logarithm holds (Theorem 3.1). The third result extends the equilibrium existence result in [2] to any finite non-zero sum game (Theorem 4.2). Before reporting these results, I present the notion of Martin-Löf [3] randomness that will be useful for future arguments. This notion of randomness is equivalent to the notion of randomness introduced in [2] but is useful for many purposes.

1 Martin-Löf randomness

Here I introduce *Martin-Löf randomness* [3] (henceforth ML randomness). This concept defines (algorithmically) random sequences in terms of statistical regularities. Its definition begins with a formulation of idealized statistical tests, defined as follows. Let X be a finite set. The set of infinite sequences $X^{\mathbb{N}}$ over X is endowed with the product topology. Any open set can be written as a union of basic sets, where a basic set has the form $N_\sigma = \{\zeta \in X^{\mathbb{N}} : \sigma \prec \zeta\}$ for some $\sigma \in X^*$.

Definition 1.1. Let X be a finite set and let $\theta \in \{0,1\}^{\mathbb{N}}$ be an oracle. Suppose that μ is a computable probability measure over $X^{\mathbb{N}}$, i.e., the mapping $\sigma \mapsto \mu(N_\sigma)$ is computable. A sequence of open sets $\{V_t\}_{t=0}^{\infty}$ is *a μ-test relative to* θ if it satisfies the following conditions:

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(1) The sequence $\{V_t\}$ is θ-effective: there is a θ-computable function $f : \mathbb{N} \to \mathbb{N} \times X^*$ such that for all $t \in \mathbb{N}$, $V_t = \bigcup\{N_\sigma : (\exists n)(f(n) = (t, \sigma))\}$.

(2) For all $t \in \mathbb{N}$, $\mu(V_t) \leq 2^{-t}$.

A sequence $\xi \in X^{\mathbb{N}}$ is *ML-random relative to* θ for μ if it passes all μ-tests relative to θ, i.e., for any μ-test $\{V_t\}_{t=0}^{\infty}$ relative to θ, $\xi \notin \bigcap_{t=0}^{\infty} V_t$.

A test $\{V_t\}_{t=0}^{\infty}$ is used to establish the statistical regularity corresponding to the complement of $\bigcap_{t=0}^{\infty} V_t$. Conditions (1) and (2) require that this test, as a sequence of open sets, be generated effectively from θ, and hence is constructive with respect to θ. A sequence is ML-random if it passes all such tests.

ML-randomness relative to θ is equivalent to θ-randomness defined in the main paper [2] using betting functions. This equivalence holds for any computable measure μ, but for the purposes here I will focus only on computable measures of the form $\mu_{\mathbf{p}}$; given a computable sequence $\mathbf{p} = (p^0, p^1, ...)$ of probability measures over X, $\mu_{\mathbf{p}}(N_\sigma) = \prod_{t=0}^{|\sigma|-1} p^t(\sigma_t)$ for any $\sigma \in X^*$. To give this equivalence result, first I shall extend the notion of betting functions to measures of the form $\mu_{\mathbf{p}}$.

Definition 1.2. Let X be a finite set and let $\mathbf{p} = (p^0, p^1, ...)$ be a computable sequence of distributions over X such that $p^t[x] > 0$ for all $x \in X$ and for all $t \in \mathbb{N}$. A function $B : X^* \to \mathbb{R}_+$ is a *betting function for* $\mu_{\mathbf{p}}$ if for all $\sigma \in X^*$, $B(\sigma) = \sum_{x \in X} p^{|\sigma|}[x] B(\sigma x)$.

For any oracle θ, a betting function B for $\mu_{\mathbf{p}}$ if B can be θ-computably approximated from below. A betting function B succeeds over a sequence $\xi \in X^{\mathbb{N}}$ if $\limsup_{n\to\infty} B(\xi[n]) = \infty$. We have the following theorem; when $|X| = 2$, this result is well-known (see Nies [4], Proposition 7.2.6).

Theorem 1.1. *Let θ be an oracle. Let $\mathbf{p}$ be a computable sequence such that $p^t[x] > 0$ for all $x \in X$ and for all $t \in \mathbb{N}$. A sequence $\xi \in X^{\mathbb{N}}$ is ML-random relative to θ for $\mu_{\mathbf{p}}$ if and only if there exists no θ-effective betting function B for $\mu_{\mathbf{p}}$ that succeeds over ξ.*

Proof. (Sketch.) ($\Rightarrow$) Suppose that $\limsup_{T\to\infty} B(\xi[T]) = \infty$ and B is a θ-effective betting function for $\mu_{\mathbf{p}}$. Let $V_t = \{\xi : (\exists s)(B(\xi[s]) > 2^t)\}$. Let $A_0 = \{\sigma \in \{0,1\}^* : B(\sigma) \geq 2^t\}$.

Enumerate elements (without repetitions) in A_0 as $A_0 = \{\sigma^1, \sigma^2, ..., \sigma^k, ...\}$. Define A_{k+1} inductively as follows: $A_{k+1} = A_k$ if there is no other $\sigma \in A_0$ such that $\sigma \prec \sigma^k$; otherwise, let $A_{k+1} = A_k - \{\sigma^k\}$. Let $A = \bigcap_{k=0}^{\infty} A_k$. Then, A is prefix-free and $V_t = \bigcup_{\sigma \in A}\{\zeta : \sigma \prec \zeta\}$. $\sum_{\sigma\in A}\prod_{t=0}^{|\sigma|-1} p^t[\sigma_t]B(\sigma) \leq B(\epsilon)$ because A is prefix-free. Therefore, $\mu_{\mathbf{p}}(V_t) = \sum_{\sigma\in A}\prod_{t=0}^{|\sigma|-1} p^t[\sigma_t] \leq 2^{-t}\sum_{\sigma\in A}\prod_{t=0}^{|\sigma|-1} p^t[\sigma_t]B(\sigma) \leq 2^{-t}$. It is routine to check that $\{V_t\}$ is θ-effective because B is. Then $\xi \in \bigcap_{t=0}^{\infty} V_t$ and hence is not $\mu_{\mathbf{p}}$-random relative to θ.

($\Leftarrow$) Suppose that $\xi \in \bigcap_{t=0}^{\infty} V_t$ for a $\mu_{\mathbf{P}}$-test $\{V_t\}$ relative to θ. Construct a betting function B as follows: Let $B^t(\sigma) = \mu_{\mathbf{p}}(V_t \cap \{\zeta : \sigma \prec \zeta\})/\mu_{\mathbf{p}}(\{\zeta : \sigma \prec \zeta\})$ and let $B = \sum_{t=0}^{\infty} B^t$. B_t is a betting function for $\mu_{\mathbf{p}}$ for each t and $B(\epsilon) = \sum_{t=0}^{\infty}\mu_{\mathbf{p}}(V_t) = 1$; so B is well-defined. B is θ-effective because $\{V_t\}$ is. $\xi \in \bigcap_{t=0}^{\infty} V_t$ implies that $\limsup_{T\to\infty} B(\xi[T]) = \infty$. □

Apparently the logic of the proof of Theorem 1.1 applies to all computable measures μ with appropriate definition of betting functions. Moreover, because the two definitions are equivalent, I will use the term θ-randomness to refer to both. Although these two definitions are equivalent, either one is more convenient for certain arguments than others. One example that illustrates the usefulness of the ML-randomness is the following proposition, which states that for any oracle θ, and for any computable probability measure μ, the set of θ-random sequences for μ has probability 1 with respect to μ. Again, the following result is well-known when $|X| = 2$ and when μ is the Lebesgue measure.

Proposition 1.1. *Suppose that X is a finite set and μ is a computable measure over $X^{\mathbb{N}}$. Then, for any oracle θ, $\mu(\{\xi \in X^{\mathbb{N}} : \xi$ is θ-random for $\mu\}) = 1$.*

Proof. Because there are only countably many θ-computable functions, the cardinality of μ-tests relative to θ is also countable. Enumerate them as $\{\{V_t^i\}_t\}_i$. The set of μ-random sequences relative to θ, denoted by $\mathrm{MLR}_{\mu}^{\theta}$, is equal to $X^{\mathbb{N}} - \bigcup_i(\bigcap_t V_t^i)$. But $\mu(\bigcap_t V_t^i) = 0$ and hence $\mu(\bigcup_i(\bigcap_t V_t^i)) = 0$. Thus, $\mu(\mathrm{MLR}_{\mu}^{\theta}) = 1$. □

While ML-randomness is useful to obtain existence, randomness based on betting functions is more useful to establish frequency results. In [2], Lemma 5.2 shows that any θ-random sequence for p has limit frequency p along any subsequence selected by a

θ-computable selection function. Here I show that the same result holds for any θ-random sequence for $\mu_{\mathbf{p}}$ with $\lim_t p^t = p$.

Theorem 1.2. *Let θ be an oracle. Suppose that ξ is θ-random for $\mu_{\mathbf{p}}$ with $p^t_x > 0$ for all $t \in \mathbb{N}$ and for all $x \in X$ and $\lim_{t\to\infty} p^t = p$. Then, for any θ-computable selection function r such that ξ^r is an infinite sequence,*

$$\lim_{T\to\infty} \sum_{t=0}^{T-1} \frac{c_x(\xi^r_t)}{T} = p[x] \text{ for all } x \in X. \tag{1}$$

Proof. Suppose, by contradiction, that there exists some $\varepsilon > 0$, some $y \in X$, and a sequence $\{T_k\}_{k=0}^{\infty}$ such that for all $k \in \mathbb{N}$, $\sum_{t=0}^{T_k-1} \frac{c_y(\xi^r_t)}{T_k} \geq p[y] + \varepsilon$. I construct a θ-effective betting function B for $\mu_{\mathbf{p}}$ that succeeds over ξ.

Let $d > 0$ be so small that $d < \frac{1}{2}\min\{p^t[y], 1 - p^t[y]\}$ for all t. Define B as follows: (a) $B(\epsilon) = 1$; (b) $B(\sigma y) = (1 + d(1 - p^{|\sigma|}[y]))B(\sigma)$ and $B(\sigma x) = (1 - dp^{|\sigma|}[y])B(\sigma)$ for all $x \neq y$ if $r(\sigma) = 1$; (c) $B(\sigma x) = B(\sigma)$ for all $x \in X$ if $r(\sigma) = 0$. Clearly, by construction, B is θ-computable because r and $\mathbf{p}$ is. B is a betting function for $\mu_{\mathbf{p}}$: If $r(\sigma) = 1$, then

$$\sum_{x\in X} p^{|\sigma|}[x]B(\sigma x) = p^{|\sigma|}[y](1 + \kappa(1 - p^{|\sigma|}[y]))B(\sigma) + \sum_{x\neq y} p^{|\sigma|}[x](1 - \kappa p^{|\sigma|}[y])B(\sigma) = B(\sigma);$$

if $r(\sigma) = 0$, then $\sum_{x\in X} p^{|\sigma|}[x]B(\sigma x) = \sum_{x\in X} p^{|\sigma|}[x]B(\sigma) = B(\sigma)$.

Now I show that $\limsup_{T\to\infty} B(\xi[t]) = \infty$. For each $k \geq 1$, define

$$D_k = \{t \leq k - 1 : r(\xi[t]) = 1,\ \xi_{t+1} = x\} \text{ and } E_k = \{t \leq k - 1 : r(\xi[t]) = 1,\ \xi_{t+1} \neq x\}.$$

Then, $B(\xi[k]) = \prod_{t\in D_k}(1 + d(1 - p^{t+1}[y]))\prod_{t\in E_k}(1 - dp^{t+1}[y])$. Let L_k be such that $\#\{0 \leq t \leq L_k - 1 : r(\xi[t]) = 1\} = T_k$, i.e., $(\xi[L_k])^r = \xi^r[T_k]$. Because ξ^r is an infinite sequence, L_k is well defined for all $k \in \mathbb{N}$. Because $\lim_{t\to\infty} p^t = p$, let T be so large that $t \geq T$ implies that $|p^t[y] - p[y]| < \delta$. Let K be the first k such that $T_k > T$. Then, for all $k > K$,

$$\begin{aligned} B(\xi[L_k]) &= \prod_{t\in D_{L_k}} (1 + d(1 - p^{t+1}[y])) \prod_{t\in E_{L_k}} (1 - dp^{t+1}[y]) \\ &\geq A\left[(1 + d(1 - p[y] - \delta))^{p[y]+\varepsilon}(1 - dp[y] - d\delta)^{1-p[y]-\varepsilon}\right]^{T_k}, \end{aligned}$$

where $A = \frac{\prod_{t\in D_{L_K}}(1+d(1-p^{t+1}[y])\prod_{t\in E_{L_K}}(1-dp^{t+1}[y]))}{(1+d(1-p[y]-\delta))^{\#D_{L_K}}(1-dp[y]-d\delta)^{\#E_{L_K}}}$. Notice that for each k, $|D_{L_k}| \geq T_k p[y] + T_k\varepsilon$. It is straightforward to verify that $(1+d(1-p[y]-\delta))^{p[y]+\varepsilon}(1-dp[y]-d\delta)^{1-p[y]-\varepsilon} > 1$ and hence $\lim_{k\to\infty} B(\xi[L_k]) = \infty$. $\square$

2 Generating random sequences

Here I give a theorem which shows that incompressible sequences can be used to compute $\mu_{\mathbf{p}}$-random sequences, for any computable sequence $\mathbf{p} = (p^0, p^1, ...)$ of probability measures over X, assuming that $p^t[x] > 0$ for all $x \in X$ and all $t \in \mathbb{N}$. The theorem generalizes the main result in Zvonkin and Levin [5], which considers the case where $X = \{0, 1\}$.

Theorem 2.1. *Let X be a finite set and let θ be an oracle. Suppose that $\eta \in \{0,1\}^{\mathbb{N}}$ is a θ-incompressible sequence. Then, there exists a θ-random sequence for $\mu_{\mathbf{p}}$ that is η-computable.*

Proof. First notice that by Lemma 5.1 in [2], there is a λ^X-random sequence $\xi' \in X^{\mathbb{N}}$ relative to θ that is η-computable, where λ^X is the uniform distribution over $X^{\mathbb{N}}$. I construct a partial computable functional $\Phi : X^{\mathbb{N}} \to X^{\mathbb{N}}$ such that if ξ' is a λ^X-random sequence, then $\Phi(\xi')$ is a $\mu_{\mathbf{p}}$-random sequence. Φ is constructed through a computable function $\phi : X^* \to X^*$ by setting Φ as $\Phi(\xi)_t = \phi(\xi[\min\{k : |\phi(\xi[k])| \geq t\}])_t$. I show that Φ satisfies the following properties:

1. Φ is well-defined over any sequence in $X^{\mathbb{N}}$ that is not computable.

2. $\lambda^X(\Phi^{-1}(A)) = \mu_{\mathbf{p}}(A)$ for any measurable A.

3. If ξ' is a λ^X-random sequence, then $\Phi(\xi')$ is a $\mu_{\mathbf{p}}$-random sequence.

(**Construction of** ϕ) ϕ is constructed through the distribution function of $\mu_{\mathbf{p}}$. Define the mapping $\Gamma : X^{\mathbb{N}} \to [0,1]$ as $\Gamma(\zeta) = \sum_{t=0}^{\infty} \iota(\zeta_t)\frac{1}{n^{t+1}}$, where $X = \{x_1, ..., x_n\}$ and $\iota(x) = i - 1$ if and only if $x = x_i$. Γ is onto but not one-to-one. However, the set $\{\zeta \in X^{\mathbb{N}} : \Gamma(\zeta) = \Gamma(\zeta') \text{ for some } \zeta' \neq \zeta\}$ is countable. Γ can be extended to X^* by setting $\Gamma(\sigma) = \sum_{t=0}^{|\sigma|-1} \frac{\iota(\sigma_t)}{n^{t+1}}$. Given Γ, the distribution function of $\mu_{\mathbf{p}}$ over $[0,1]$ is defined by $g : [0,1] \to [0,1]$ as $g(r) = \mu_{\mathbf{p}}(\{\zeta : \Gamma(\zeta) \le r\})$. Define $h = g^{-1}$; h exists because $\mu_{\mathbf{p}}$ has no atoms. Therefore, $r \le g(s)$ if and only if $h(r) \le s$. Hence, $\mu_{\mathbf{p}}(\Gamma^{-1}([0,r])) = g(r) = \lambda^X(\Gamma^{-1}([0,g(r)])) = \lambda^X(\Gamma^{-1}(h^{-1}([0,r])))$.

Now I construct the function ϕ using the distribution function g. The idea of construction is the following: Because g is continuous, any open interval has a pre-image that is also open. Because each finite sequence in X^* can be regarded as an open interval, it can be mapped into another via g. As the length of the finite sequence increases, the interval shrinks and finally the functional Φ obtains.

Let ϵ be the empty string. Define $g^0(\epsilon) = 0$ and $g^1(\epsilon) = 1$. For $\tau \in X^* - \{\epsilon\}$, define $g^0(\tau) = \sum\{\mu_{\mathbf{p}}(N_\sigma) : \Gamma(\sigma) \le \Gamma(\tau) - \frac{1}{n^{|\tau|}}, |\sigma| = |\tau|\}$ and $g^1(\tau) = \sum\{\mu_{\mathbf{p}}(N_\sigma) : \Gamma(\sigma) \le \Gamma(\tau), |\sigma| = |\tau|\}$. For any $\zeta \in X^{\mathbb{N}}$, $\Gamma(\zeta) \le \Gamma(\tau)$ if and only if $\Gamma(\zeta[|\tau|]) \le \Gamma(\tau) - \frac{1}{n^{|\tau|}}$ or $\Gamma(\zeta) = \Gamma(\tau)$, and $\Gamma(\zeta) \le \Gamma(\tau) + \frac{1}{n^{|\tau|}}$ if and only if $\Gamma(\zeta[|\tau|]) \le \Gamma(\tau)$ or $\Gamma(\zeta) = \Gamma(\tau) + \frac{1}{n^{|\tau|}}$. Because $\mu_{\mathbf{p}}$ has no atoms, $g^0(\tau) = g(\Gamma(\tau))$ and $g^1(\tau) = g(\Gamma(\tau) + \frac{1}{n^{|\tau|}})$. Therefore, for each $t > 0$, the class of intervals $\{[g^0(\tau), g^1(\tau)] : \tau \in X^*, |\tau| = t\}$ forms a partition of $[0,1]$.

Construct ϕ as follows: given a string $\sigma \in X^*$ (recall that $|X| = n$), let

$$a_\sigma = \Gamma(\sigma) \text{ and } b_\sigma = \Gamma(\sigma) + \frac{1}{n^{|\sigma|}}.$$

Let $\phi(\sigma)$ be the longest τ with $|\tau| \le |\sigma|$ such that $[a_\sigma, b_\sigma] \subset [g^0(\tau), g^1(\tau)]$. $\phi(\sigma)$ is well-defined, because the intervals $[g^0(\tau), g^1(\tau)]$ with $|\tau| = t$ form a partition and $[g^0(\epsilon), g^1(\epsilon)] = [0,1]$. ϕ is computable. Define Φ as $\Phi(\xi)_t = \phi(\xi[\min\{k : |\phi(\xi[k])| \ge t\}])_t$.

(**Φ satisfies property 1.**) Φ is well defined if $\lim_{t\to\infty} \phi(\xi[t]) = \infty$ and it outputs a finite sequence otherwise (i.e., for large t's, $\Phi(\xi)_t$ is not defined). First I show that ϕ satisfies that for any $\sigma, \tau \in X^*$, $\sigma \subset \tau$ implies $\phi(\sigma) \subset \phi(\tau)$. Suppose that $\sigma \subset \sigma'$ and $\tau = \phi(\sigma)$, $\tau' = \phi(\sigma')$. It is easy to check that $a_\sigma \le a_{\sigma'}$ and $b_{\sigma'} \le b_\sigma$. Now, if $\Gamma(\tau') \ge \Gamma(\tau) + \frac{1}{n^{|\tau|}}$, then $a_{\sigma'} \ge g^0(\tau') = g(\Gamma(\tau')) \ge g(\Gamma(\tau) + \frac{1}{n^{|\tau|}}) = g^1(\tau) \ge b_\sigma \ge b_{\sigma'}$, a contradiction to $a_\sigma < b_\sigma$. Hence, $\Gamma(\tau') < \Gamma(\tau) + \frac{1}{n^{|\tau|}}$. By construction of ϕ, $|\tau'| \ge |\tau|$. If $\Gamma(\tau') < \Gamma(\tau)$, then $\Gamma(\tau') \le \Gamma(\tau) - \frac{1}{n^{|\tau|}}$, and hence, $b_\sigma \le b_{\sigma'} \le g^1(\tau') = g(\Gamma(\tau') + \frac{1}{n^{|\tau'|}}) \le g(\Gamma(\tau)) = g^0(\tau) \le a_\sigma$, a contradiction to $a_\sigma < b_\sigma$. Therefore, $\Gamma(\tau) \le \Gamma(\tau') < \Gamma(\tau) + \frac{1}{n^{|\tau|}}$ and so $\tau \subset \tau'$.

Then I show that, for any sequence ζ such that $h(\Gamma(\zeta)) \ne \frac{m}{n^t}$ for any $m, n, t \in \mathbb{N}$ (recall that $h = g^{-1}$), $\lim_{t\to\infty} \phi(\zeta[t]) = \infty$. Consider any such ζ. For any given K, there exists some $l \in \mathbb{N}$ such that $h(\Gamma(\zeta)) \in (\frac{l}{n^K}, \frac{l+1}{n^K})$. Let $\varepsilon = \min\{h(\Gamma(\zeta)) - \frac{l}{n^K}, \frac{l+1}{n^K} - h(\Gamma(\zeta))\}$. Because h is continuous, there is some T such that $t \geq T$ implies that

$$\min\{|h(b_{\zeta[t]}) - h(\Gamma(\zeta))|, |h(\Gamma(\zeta) - h(a_{\zeta[t]})|\} \leq \frac{\varepsilon}{2} \text{ and so } [h(a_{\zeta[t]}), h(b_{\zeta[t]})] \subseteq (\frac{l}{n^K}, \frac{l+1}{n^K}).$$

Thus, if $t \geq \max\{T, K\}$, then $[a_{\zeta[t]}, b_{\zeta[t]}] \subset [g(\frac{l}{n^K}), g(\frac{l+1}{n^K})] = [g^0(\frac{l}{n^K}), g^1(\frac{l}{n^K})]$, and so $|\phi(\zeta[t])| \geq K$. Clearly, any sequence ζ that satisfies $h(\Gamma(\zeta)) = \frac{m}{n^t}$ for some $m, n, t \in \mathbb{N}$ is computable, and so if $\Phi(\zeta)$ is not well-defined, ζ is computable.

(Φ **satisfies property 2.**) I first claim that if Φ is well-defined over ζ (the set of such ζ's is denoted by $D(\phi)$), then $\Gamma(\Phi(\zeta)) = h(\Gamma(\zeta))$. Let ε be given, and let K be so large that $\varepsilon < \frac{1}{n^{K-1}}$. Since $\zeta \in D(\phi)$, there exists T such that $t \geq T$ implies that $|\phi(\zeta[t])| \geq K$. Then, for all $t \geq T$, $h(\Gamma(\zeta)) \in [h(a_{\zeta[t]}), h(b_{\zeta[t]})] \subseteq [a_{\phi(\zeta[t])}, b_{\phi(\zeta[t])}]$, and so $h(\Gamma(\zeta)) - \Gamma(\phi(\zeta[t])) \leq \frac{1}{n^K} \leq \varepsilon$. Thus, $\Gamma(\Phi(\zeta)) = \lim_{t\to\infty} \Gamma(\phi(\zeta[t])) = h(\Gamma(\zeta))$. Moreover, for almost all $r \in [0,1]$ (except for countably many of them), there is a sequence $\zeta \in X^{\mathbb{N}}$ such that $\Gamma(\Phi(\zeta)) = r$, because h is strictly increasing and is continuous. Also,

$$\Gamma(\Phi(\zeta)) \geq \Gamma(\Phi(\zeta')) \Leftrightarrow \Gamma(\zeta) \geq \Gamma(\zeta'). \tag{2}$$

I show that $\lambda^X_\Phi = \mu_{\mathbf{p}}$ by demonstrating that they share the same distribution function g, where $\lambda^X_\Phi(A) = \lambda^X(\Phi^{-1}(A))$: for any ζ^*,

$$\lambda^X_\Phi(\{\zeta : \Gamma(\zeta) \leq \Gamma(\Phi(\zeta^*))\}) = \lambda^X(\{\zeta : \Gamma(\Phi(\zeta)) \leq \Gamma(\Phi(\zeta^*))\})$$

$$= \lambda^X(\{\zeta : \Gamma(\zeta) \leq \Gamma(\zeta^*)\}) = \Gamma(\zeta^*) = g(\Gamma(\Phi(\zeta^*))).$$

(Recall that, for all but a countable set of numbers $r \in [0,1]$, there is a ζ^* such that $\Gamma(\Phi(\zeta^*)) = r$. The gaps may be filled by assigning arbitrary values on Φ when it is not well-defined. The first equality comes from the definition of λ^X_Φ and the second comes from equation (2).)

(Φ **satisfies property 3.**) Recall that Φ is well-defined over any incomputable sequence. Thus, if ξ' is θ-random for λ^X, $\xi' \in D(\phi)$. Let $\zeta' = \Phi(\xi')$. Now I show that ζ' is θ-random for $\mu_{\mathbf{p}}$. Suppose not, and suppose that there is a $\mu_{\mathbf{p}}$-test $\{V_t\}_{t=0}^{\infty}$ relative to θ such that

$\zeta' \in \bigcap_{t=0}^{\infty} V_t$. Let $U_t = \{\xi : (\exists \zeta \in V_t)\zeta = \Phi(\xi)\}$. Because ϕ is computable, $\{U_t\}_{t=0}^{\infty}$ is θ-effective. Moreover, $\lambda^X(U_t) = \lambda^X(\Phi^{-1}(V_t)) = \mu_{\mathbf{p}}(V_t) \leq \frac{1}{2^t}$. Therefore, $\{U_t\}_{t=0}^{\infty}$ is a λ^X-test relative to θ. But $\xi' \in \bigcap_{t=0}^{\infty} U_t$ because $\zeta' \in \bigcap_{t=0}^{\infty} V_t$, a contradiction. Since ϕ is computable, ζ' is ξ'-computable and hence is η-computable. □

3 Law of the Iterated Logarithm

Here I give a general Law of the Iterated Logarithm that is satisfied by any ML-random sequence for $\mu_{\mathbf{p}}$.

Theorem 3.1. *Suppose that ξ is a ML-random sequence for $\mu_{\mathbf{p}}$ with $\mathbf{p} = (p^0, p^1, ..., p^t, ...)$. Then, for any $x \in X$,*

$$\limsup_{T\to\infty} \frac{|\sum_{t=0}^{T-1}(c_x(\xi_t) - p^t[x])|}{\sqrt{2(\sum_{t=0}^{T-1} p^t[x](1-p^t[x])) \log\log \sqrt{(\sum_{t=0}^{T-1} p^t[x](1-p^t[x]))}}} = 1, \tag{3}$$

Proof. The positive part of equation (3) is equivalent to the following two conditions:

(a) for all rational $\varepsilon > 0$,

$$(\exists S)(\forall T \geq S) \sum_{t=0}^{T-1}(c_x(\xi_t) - p^t[x]) \leq \sqrt{2(1+\varepsilon)(\sum_{t=0}^{T-1} p^t[x](1-p^t[x])) \log\log \sqrt{(\sum_{t=0}^{T-1} p^t[x](1-p^t[x]))}}.$$

(b) for all rational $\varepsilon > 0$,

$$(\forall S)(\exists T \geq S) \sum_{t=0}^{T-1}(c_x(\xi_t) - p^t[x]) \geq \sqrt{2(1-\varepsilon)(\sum_{t=0}^{T-1} p^t[x](1-p^t[x])) \log\log \sqrt{(\sum_{t=0}^{T-1} p^t[x](1-p^t[x]))}}.$$

I show that (a) and (b) hold and the negative part is completely symmetric. Let

$$E_T^{\varepsilon} = \left\{ \zeta : \sum_{t=0}^{T-1}(c_x(\zeta_t) - p^t[x]) > \sqrt{2(1+\varepsilon)(\sum_{t=0}^{T-1} p^t[x](1-p^t[x])) \log\log \sqrt{(\sum_{t=0}^{T-1} p^t[x](1-p^t[x]))}} \right\},$$

and

$$F_T^\varepsilon = \left\{ \zeta : \sum_{t=0}^{T-1} (c_x(\zeta_t) - p^t[x]) < \sqrt{2(1-\varepsilon)(\sum_{t=0}^{T-1} p^t[x](1-p^t[x])) \log\log \sqrt{(\sum_{t=0}^{T-1} p^t[x](1-p^t[x]))}} \right\}.$$

Clearly, condition (a) is equivalent to $\xi \notin \bigcap_{S=0}^\infty \bigcup_{T=S}^\infty E_T^\varepsilon$ and condition (b) is equivalent to $\xi \notin \bigcup_{S=0}^\infty \bigcap_{T=S}^\infty F_T^\varepsilon$. By Theorem 7.5.1 in Chung [1],

$$\mu_{\mathbf{p}}(\bigcap_{S=0}^\infty \bigcup_{T=S}^\infty E_T^\varepsilon) = 0 \text{ and } \mu_{\mathbf{p}}(\bigcup_{S=0}^\infty \bigcap_{T=S}^\infty F_T^\varepsilon) = 0.$$

It then follows that $\mu_{\mathbf{p}}(\bigcap_{T=S}^\infty F_T^\varepsilon) = 0$ for any $S \in \mathbb{N}$. Because F_T^ε is computable (uniformly in T), $\{F_T^\varepsilon\}_{T=S}^\infty$ is a $\mu_{\mathbf{p}}$-test for any S (notice that $\mu_{\mathbf{p}}(F_T^\varepsilon)$ is also computable). Therefore, $\xi \notin \bigcup_{S=0}^\infty \bigcap_{T=S}^\infty F_T^\varepsilon$. This proves (b).

On the other hand, the set E_T^ε is computable (uniformly in T) and so the sets $\{\bigcup_{T=S}^\infty E_T^\varepsilon\}_{S\in\mathbb{N}}$ is effective. For $\{\bigcup_{T=S}^\infty E_T^\varepsilon\}_{S=0}^\infty$ to be a test, we need to show that $\mu_{\mathbf{p}}(\bigcup_{T=S}^\infty E_T^\varepsilon)$ has a computable upper bound for all S. From the proof in Theorem 7.5.1 in Chung [1], we know that there exists a constant $A > 0$ and a number $\overline{k} > 0$ such that for all $k \geq \overline{k}$ (with the provision that $c^2(1+\frac{\varepsilon}{2}) < 1+\varepsilon$), c.f. p. 216),

$$\mu_{\mathbf{p}}(\bigcup_{T=T_k}^{T_{k+1}-1} E_T^\varepsilon) < \frac{A}{(k\log c)^{1+\frac{\varepsilon}{2}}},$$

where $T_k = \max\{T : \sqrt{\sum_{t=0}^T p^t[x](1-p^t[x])} \leq c^k\}$ and $c = 1 + \frac{\varepsilon}{10}$ (for ε small enough, $c^2(1+\frac{\varepsilon}{2}) < 1+\varepsilon$).

Let's define $G_0 = \bigcup_{T=0}^{T_1-1} E_T^\varepsilon$ and $G_k = \bigcup_{T=T_k}^{T_{k+1}-1} E_T^\varepsilon$ for $k > 0$. Clearly,

$$\bigcap_{S=0}^\infty \bigcup_{k=S}^\infty G_k = \bigcap_{S=0}^\infty \bigcup_{T=S}^\infty E_T^\varepsilon.$$

Now, because T_k is a computable function of k, $\{\bigcup_{k=S}^\infty G_k\}_{S=0}^\infty$ is also an effective sequence open sets. I now show that there is a computable mapping $i \mapsto S_i$ so that $\mu_{\mathbf{p}}(\bigcup_{k=S_i}^\infty G_k) \leq \frac{1}{2^i}$. It is easy to verify that

$$\sum_{k=S}^\infty \frac{A}{(k\log c)^{1+\frac{\varepsilon}{2}}} \leq \int_{a=S-1}^\infty \frac{A}{(a\log c)^{1+\frac{\varepsilon}{2}}} = A(S-1)^{-\frac{\varepsilon}{2}}(\log c)^{-1-\frac{\varepsilon}{2}}.$$

Let $B \in \mathbb{N}$ be such that $B > (A(\log c)^{-1-\frac{\varepsilon}{2}})^{\frac{2}{\varepsilon}}$ and let $N \in \mathbb{N}$ be such that $N > \frac{2}{\varepsilon}$. Take $S_i = B2^{Ni} + 1$, and it follows that $\mu_{\mathbf{p}}(\bigcup_{k=S_i}^{\infty} G_k) \leq \frac{1}{2^i}$. This shows that $\{\bigcup_{k=S}^{\infty} G_k\}_{S=0}^{\infty}$ is a $\mu_{\mathbf{p}}$-test, and so $\xi \notin \bigcap_{S=0}^{\infty} \bigcup_{k=S}^{\infty} G_k = \bigcap_{S=0}^{\infty} \bigcup_{T=S}^{\infty} E_T^{\varepsilon}$. This proves (a). $\square$

As a corollary, for any $p \in \Delta(X)$ and any ML-random sequence ξ for μ_p,

$$\limsup_{T\to\infty} \frac{|\sum_{t=0}^{T} c_x(\xi_t) - Tp[x]|}{\sqrt{2p[x](1-p[x])T \log\log T}} = 1. \tag{4}$$

Notice that (4) follows from (3) by taking $p^t = p$ for all $t \in \mathbb{N}$. However, if ξ is ML-random for $\mu_{\mathbf{p}}$ such that $\lim_{t\to\infty} p^t = p$, (4) may not hold for ξ, even though Theorem 1.2 implies that the frequency condition (1) holds for ξ for any selection function r. Indeed, the following theorem shows that, for any θ-incompressible sequence η, there is a η-computable sequence ξ that satisfies (1) but fails (4).

Theorem 3.2. *Suppose that η is θ-incompressible. For any non-degenerate $p \in \Delta(X)$, there exists a η-computable sequence ξ such that*
(a) it satisfies the convergence requirement (1) for any θ-computable selection function r so that ξ^r is an infinite sequence;
(b) for some $y \in X$ with $p[y] \in (0,1)$,

$$\lim_{T\to\infty} \frac{\sum_{n=0}^{T} c_y(\xi_n) - Tp[y]}{\sqrt{2p[y](1-p[y])T \log\log T}} = \infty. \tag{5}$$

Proof. Let y, y' be such that $p[y] \in (0,1)$ and $p[y'] \in (0,1)$. For any real number s, let $\llcorner s \lrcorner$ be the largest integer no greater than s. Construct the sequence $\mathbf{p} = (p^0, p^1, ...)$ as follows ($\bar{t}$ is the smallest t such that $\llcorner t^{0.4} \lrcorner > \frac{1}{p[x]}$):

(a) $p^t[x] = p[x]$ if $x \neq y$ and $x \neq y'$;

(b) $p^t[y] = p[y]$ if $t \leq \bar{t}$ and $p^t[y] = p[y] - \frac{1}{\llcorner t^{0.4} \lrcorner}$ otherwise;

(c) $p^t[y'] = p[y']$ if $t \leq \bar{t}$ and $p^t[y'] = p[y'] + \frac{1}{\llcorner t^{0.4} \lrcorner}$ otherwise.

By construction, $p^t[x] = 0$ if and only if $p[x] = 0$, and $\lim_{t\to\infty} p^t = p$. $\mathbf{p}$ is computable. By Theorem 2.1, there is a θ-random sequence ξ for $\mu_{\mathbf{p}}$ that is η-computable. Now, let $X_0 = \{x \in X : p[x] > 0\}$, then the sequence ξ that is θ-random for $\mu_{\mathbf{p}}$ can be regarded as a sequence in $X_0^{\mathbb{N}}$. Theorem 1.2 implies that ξ satisfies the convergence requirement (1).

By Theorem 3.1,

$$\limsup_{T\to\infty} \frac{|\sum_{t=0}^{T-1}(c_y(\xi_t) - p^t[y])|}{\sqrt{2(\sum_{t=0}^{T-1} p^t[y](1-p^t[y]))\log\log\sqrt{(}\sum_{t=0}^{T-1} p^t[y](1-p^t[y]))}} = 1. \tag{6}$$

For any $T > \bar{t}$, $\frac{\sum_{t=0}^{T-1} c_y(\xi_t) - Tp[y]}{\sqrt{2Tp[y](1-p[y])\log\log T}} = \frac{\sum_{t=0}^{T-1}(c_y(\xi_t) - p^t[y])}{\sqrt{2Tp[y](1-p[y])\log\log T}} + \frac{\sum_{t=\bar{t}+1}^{T-1}\frac{1}{\lfloor t^{0.4}\rfloor}}{\sqrt{2Tp[y](1-p[y])\log\log T}}$.

I claim that

$$\lim_{T\to\infty} \frac{\sum_{t=\bar{t}+1}^{T-1}\frac{1}{\lfloor t^{0.4}\rfloor}}{\sqrt{2Tp[y](1-p[y])\log\log T}} = \infty; \tag{7}$$

and there exists some $B > 0$ such that for all T large enough,

$$\frac{|\sum_{t=0}^{T-1}(c_y(\xi_t) - p^t[y])|}{\sqrt{2Tp[y](1-p[y])\log\log T}} < B. \tag{8}$$

The theorem follows directly from (7) and (8).

Now I prove the claim. For all t, $\lfloor t^{0.4}\rfloor \le t^{0.4}$ and so $\frac{1}{t^{0.4}} \le \frac{1}{\lfloor t^{0.4}\rfloor}$. Then, $\sum_{t=1}^{T-1}\frac{1}{\lfloor t^{0.4}\rfloor} \ge \sum_{t=1}^{T-1}\frac{1}{t^{0.4}} \ge \int_{a=1}^{T-1} a^{-0.4}da - 1 \ge (T-1)^{0.6} - 2$. Therefore, for T large enough,

$$\frac{\sum_{t=\bar{t}+1}^{T-1}\frac{1}{\lfloor t^{0.4}\rfloor}}{\sqrt{2Tp[y](1-p[y])\log\log T}} \ge \frac{0.5T^{0.6}}{\sqrt{2Tp[y](1-p[y])\log\log T}} = C\frac{T^{0.1}}{\sqrt{\log\log T}} \tag{9}$$

for some constant $C > 0$. Because $\lim_{T\to\infty}\frac{T^{0.1}}{\sqrt{\log\log T}} = \infty$, (9) implies (7).

Because of (6), to prove (8), it suffices to show that for T large enough,

$$\frac{\sqrt{2(\sum_{t=0}^{T-1} p^t[y](1-p^t[y]))\log\log\sqrt{(}\sum_{t=0}^{T-1} p^t[y](1-p^t[y]))}}{\sqrt{2Tp[y](1-p[y])\log\log T}} \tag{10}$$

is bounded. Now, for T large enough,

$$\sum_{t=0}^{T-1} p^t[y](1-p^t[y])) = Tp[y](1-p[y]) + (2p[y]-1)\sum_{t=\bar{t}+1}^{T-1}\frac{1}{\lfloor t^{0.4}\rfloor} - \sum_{t=\bar{t}+1}^{T-1}(\frac{1}{\lfloor t^{0.4}\rfloor})^2. \tag{11}$$

Because for t large enough, $\frac{1}{2}t^{0.4} < \lfloor t^{0.4}\rfloor$, there is a constant $A > 0$ such that $\sum_{t=\bar{t}+1}^{T-1}\frac{1}{\lfloor t^{0.4}\rfloor} < \sum_{t=\bar{t}+1}^{T-1}\frac{2}{t^{0.4}} + A < 2T^{0.6} + A$. Similarly, there is a constant $A' > 0$ such that $\sum_{t=\bar{t}+1}^{T-1}(\frac{1}{\lfloor t^{0.4}\rfloor})^2 < \sum_{t=\bar{t}+1}^{T-1}\frac{4}{t^{0.8}} + A' < 4T^{0.2} + A'$. Hence, $\frac{\sum_{t=0}^{T-1} p^t[y](1-p^t[y])}{Tp[y](1-p[y])} < 1 + \frac{2|2p[y]-1|}{T^{0.4}p[y](1-p[y])} + \frac{|2p[y]-1|A + A'}{Tp[y](1-p[y])} + \frac{4}{T^{0.8}p[y](1-p[y])}$, and so, for T large enough, $\frac{\sum_{t=0}^{T-1} p^t[y](1-p^t[y]))}{Tp[y](1-p[y])} < 2$. Equation (11) also implies that, for T large enough,

$$\sum_{t=0}^{T-1} p^t[y](1-p^t[y]) \le (2|2p[y]-1| + 5 + p[y](1-p[y]))T = A''T,$$

and hence

$$\frac{\log\log\sqrt{(\sum_{t=0}^{T-1}p^t[y](1-p^t[y]))}}{\log\log T}\le\frac{\log(\log T+\log A'')}{\log\log T}\le\frac{\log 2+\log\log T}{\log\log T}.$$

So for T large enough, $\frac{\log\log\sqrt{(\sum_{t=0}^{T-1}p^t[y](1-p^t[y]))}}{\log\log T}\le 2$. Thus, the expression in (10) is bounded by 2, and this proves (8). □

4 Extensions to N-person games

Here I show that the existence result in [2] can be extended to N-person games. Let $g=\langle(X_1,...,X_N),(h_1,...,h_N)\rangle$ be a finite N-person normal-form game, where X_i is the set of actions and h_i is the payoff function for player i. In the repeated game with complexity constraints, each player i is endowed with an oracle $\theta^i\in\{0,1\}^{\mathbb{N}}$ to implement his strategy with an oracle program. Hence, a strategy is feasible for player i if and only if it is θ^i-computable. The set of all θ^i-computable total functions is denoted by $\mathcal{C}(\theta^i)$.

Definition 4.1. Let $g=\langle(X_1,...,X_N),(h_1,...,h_N)\rangle$ be a finite game and let $(\theta^1,\ \theta^2,\ ...,\ \theta^N)$ be N oracles. The *repeated game with oracles* $(\theta^1,\ \theta^2,\ ...,\ \theta^N)$ *based on* g, denoted by $RG(g,\theta^1,...,\theta^N)$, is a tuple $\langle(\mathcal{A}_1,...,\mathcal{A}_N),(u_1,...,u_N)\rangle$ such that

(a) $\mathcal{A}_i=\{\alpha_i : X_{-i}^*\to X_i : \alpha_i\in\mathcal{C}(\theta^i)\}$ is the set of player i's strategies;

(b) $u_i:\mathcal{A}_1\times...\times\mathcal{A}_N\to\mathbb{R}$ is player i's payoff function defined as

$$u_i(\alpha_1,...,\alpha_N)=\liminf_{T\to\infty}\sum_{t=0}^{T-1}\frac{h_i(\xi_t^{\alpha,1},...,\xi_t^{\alpha,N})}{T},\tag{12}$$

where $(\xi_t^{\alpha,1},...,\xi_t^{\alpha,N})$ is the outcome of period t for the strategy profile $\alpha=(\alpha_1,...,\alpha_N)$ defined by $\xi_0^{\alpha,j}=\alpha_j(\epsilon)$ and for any $t\ge 0$, $\xi_{t+1}^{\alpha,j}=\alpha_j(\xi_0^{\alpha,-j},\xi_1^{\alpha,-j},...,\xi_t^{\alpha,-j})$, for all $j=1,...,N$.

The notion of mutual complexity can be extended to the N-player case. For any finite collection of oracles $(\theta^1,...,\theta^N)$, define the product oracle $\bigotimes_{i=1}^N\theta^i=\theta^1\otimes\theta^2\otimes...\otimes\theta^N$ as $(\bigotimes_{i=1}^N\theta^i)_t=(\theta_t^1,...,\theta_t^N)$ for all $t\in\mathbb{N}$. We say that the oracles $(\theta^1,...,\theta^N)$ are *mutually complex* if for each $i=1,...,N$, θ^i is $\bigotimes_{j\neq i}\theta^j$-incompressible. As in the 2-player case, mutual complexity is a generic property for the N-player case as well. Theorem 2.1 shows that, under mutual complexity, for each i and any $p \in \Delta(X_i)$, there exists a θ^i-computable sequence ξ^i that is $\bigotimes_{j\neq i}\theta^j$-random for μ_p. However, equilibrium existence requires $\bigotimes_{j\neq i}\xi^j$ to be θ^i-random for $\mu_{\otimes_{j\neq i}p^j}$; for a given collection of distributions $(p^1,...,p^N)\in\Delta(X_1)\times...\times\Delta(X_N)$, $\mu_{\otimes_{i=1}^N p^i}\in\Delta(X_1\times...\times X_N)$ is the i.i.d. measure generated by $\otimes_{i=1}^N p^i$ ($\otimes_{i=1}^N p^i[(x_1,..,x_N)]=\prod_{i=1}^N p^i[x_i]$). To this end I give another theorem.

Theorem 4.1. *Let X and Y be two finite sets and let $\theta\in\{0,1\}^{\mathbb{N}}$ be an oracle. Suppose that $p\in\Delta(X)$ and $q\in\Delta(Y)$.*

(a) If $\xi\otimes\zeta\in(X\times Y)^{\mathbb{N}}$ is θ-random $\mu_{p\otimes q}$, then ξ is $\zeta\otimes\theta$-random for μ_p.

(b) If $\xi\in X^{\mathbb{N}}$ is $\zeta\otimes\theta$-random for μ_p and ζ is random for μ_q, then $\xi\otimes\zeta$ is θ-random for $\mu_{p\otimes q}$.

Proof. (a) Suppose that ξ is not $\zeta\otimes\theta$-random for μ_p. Then $\xi\in\bigcap_{t=0}^{\infty}V_t$ for a $\zeta\otimes\theta$-effective sequence of sets $\{V_t\}_{t=0}^{\infty}$ in $X^{\mathbb{N}}$ such that $\mu_p(V_t)\le\frac{1}{2^t}$. By the Enumeration Theorem, there exists an effective enumeration of all oracle machines, $\{\varphi_1^{\eta\otimes\theta},\varphi_2^{\eta\otimes\theta},...,\varphi_k^{\eta\otimes\theta},...\}$, such that (1) the function $U(k,n)=\varphi_k^{\eta\otimes\theta}(n)$ is $\eta\otimes\theta$-computable for any $\eta\in Y^{\mathbb{N}}$; (2) for each k, the function $\varphi_k^{\eta\otimes\theta}(\cdot)$ can be thought of as a functional from $(Y\times\{0,1\})^{\mathbb{N}}$ to $\mathbb{N}^{\mathbb{N}}$, and it is computable in the sense that the function $h(\sigma,n)=\varphi_k^{\sigma}(n)$ is computable, where $\varphi_k^{\sigma}(n)=m$ if the k-th oracle machine halts within $|\sigma|$ steps and using information only contained in $\sigma\in(Y\times\{0,1\})^*$ and h is undefined otherwise.

Now, there exists a number k such that $f=\varphi_k^{\zeta\otimes\theta}$. For any $\eta\in Y^{\mathbb{N}}$, define $U_{t,s}^{\eta}$ by taking $\xi'\in U_{t,s}^{\eta}$ if and only if for some $\sigma\prec\xi'$,

$$(\exists n<s)\varphi_k^{(\eta\otimes\theta)[s]}(n)=(t,\sigma) \tag{13}$$

(notice that $U_{t,s}^{\eta}$ only depends on $\eta[s]$, that is, the first s elements of η) and let $U_t^{\eta}=\bigcup_{\mu_p(U_{t,s}^{\eta})\le\frac{1}{2^t}}U_{t,s}^{\eta}$. For each s and t, the set $U_{t,s}^{\eta}$ can be written as a union of finite basic sets and those basic sets are uniformly computable in s and t. Thus, the sequence $\{U_t^{\eta}\}_{t\in\mathbb{N}}$ is $\eta\otimes\theta$-effective. Let $V_t=\{\xi'\otimes\eta\in(X\times Y)^{\mathbb{N}}:\xi'\in U_t^{\eta}\}$. Then $\xi'\otimes\eta\in V_t$ if and only if for some $\sigma\otimes\tau\prec\xi'\otimes\eta$ such that $\mu_p(U_{t,s}^{\tau})\le\frac{1}{2^t}$ and (13) holds for σ and $\eta[s]=\tau[s]$. Thus,

$\{V_t\}_{t=0}^{\infty}$ is θ-effective. Now I show that $\mu_{p\otimes q}(V_t) \leq \frac{1}{2^t}$:

$$\mu_{p\otimes q}(V_t) = \int_{(X\times Y)^{\mathbb{N}}} \chi_{V_t}(\xi' \otimes \eta) d\mu_{p\otimes q}(\xi' \otimes \eta)$$
$$= \int_{Y^{\mathbb{N}}} \int_{X^{\mathbb{N}}} \chi_{U_t^{\eta}}(\xi') d\mu_p(\xi') d\mu_q(\eta) = \int_{Y^{\mathbb{N}}} \mu_p(U_t^{\eta}) d\mu_q(\eta) \leq \frac{1}{2^t}.$$

Thus, $\{V_t\}_{t=0}^{\infty}$ is a θ-effective $\mu_{p\otimes q}$-test. But $\xi \otimes \zeta \in V_t$ for all $t \in \mathbb{N}$, and so $\xi \otimes \zeta$ is not θ-random for $\mu_{p\otimes q}$.

(b) Suppose that $\xi\otimes\zeta \in (X\times Y)^{\mathbb{N}}$ is not θ-random for $\mu_{p\otimes q}$. Then, $\xi\otimes\zeta \in \bigcap_{t=0}^{\infty} U_t$ for some θ-effective test $\{U_t\}$ in $(X \times Y)^{\mathbb{N}}$ such that $\mu_{p\otimes q}(U_t) \leq \frac{1}{4^t}$. Suppose that the θ-effective function f, $\xi' \otimes \zeta' \in U_t$ if and only if for some $\sigma \in (X \times Y)^*$ and for some n, $f(n) = (t, \sigma)$. Let $f = \varphi_k^{\theta}$. Define $V_t^{\zeta'} = \{\xi' \in X^{\mathbb{N}} : \xi' \otimes \zeta' \in U_t\}$ and $W_t = \{\zeta' \in Y^{\mathbb{N}} : \mu_p(V_t^{\zeta'}) > \frac{1}{2^t}\}$. A similar argument as that in (a) shows that for each $\zeta' \in Y^{\mathbb{N}}$, $\{V_t^{\zeta'}\}_{t=0}^{\infty}$ is $\zeta' \otimes \theta$-effective and $\{W_t\}$ is θ-effective. Now I show that $\mu_q(W_t) \leq \frac{1}{2^t}$:

$$\mu_{p\otimes q}(U_t) = \int_{(X\times Y)^{\mathbb{N}}} \chi_{U_t}(\xi' \otimes \zeta') d\mu_{p\otimes q}(\xi' \otimes \zeta') = \int_{(X\times Y)^{\mathbb{N}}} \chi_{V_t^{\zeta'}}(\xi') d\mu_{p\otimes q}(\xi' \otimes \zeta')$$
$$= \int_{Y^{\mathbb{N}}} \mu_p(V_t^{\zeta'}) d\mu_q(\zeta') > \int_{Y^{\mathbb{N}}} \frac{1}{2^t} \chi_{W_t}(\zeta') d\mu_q(\zeta') = \frac{1}{2^t} \mu_q(W_t).$$

Thus, $\mu_q(W_t) < 2^t \mu_{p\otimes q}(U_t) \leq \frac{1}{2^t}$.

Because ζ is q-random relative to θ, by Solovay's Theorem (Nies [4], Proposition 3.2.19), there is some $L \in \mathbb{N}$ such that $\zeta \notin W_t$ for all $t \geq L$. Thus, by construction, for all $t \geq L$, $\mu_p(V_t^{\zeta}) \leq \frac{1}{2^t}$. But $\xi \in V_t^{\zeta}$ for all t, and so ξ is not $\zeta \otimes \theta$-random for μ_p. □

The following theorem shows that mutual complexity implies equilibrium existence.

Theorem 4.2 (Existence)**.** *Suppose that the oracles $(\theta^1, ..., \theta^N)$ satisfy mutual complexity. For any mixed equilibrium $p = (p^1, ..., p^N)$ of g, there exists a Nash equilibrium of $RG(g, \theta^1, ..., \theta^N)$, consisting of history-independent strategies $(\xi^1, ..., \xi^N)$, such that the equilibrium payoff for player i is $h_i(p)$ and the limit frequency of ξ^i is p^i.*

Proof. By Theorem 2.1, for each $i = 1, ..., N$, there exists a sequence $\xi^i \in X_i^{\mathbb{N}}$ that is θ^i-computable and is $\bigotimes_{j\neq i} \theta^j$-random for μ_{p^i}. I now show that $(\xi^1, ..., \xi^N)$ is a Nash equilibrium. It suffices to show that $u_i(\xi^i;\xi^{-i}) = h_i(p)$ and for all $\alpha_i \in \mathcal{A}_i$, $u_i(\alpha_i;\xi^{-i}) \leq h_i(p)$, that is, for all $\alpha_i \in \mathcal{A}_i$,

$$\liminf_{T\to\infty} \sum_{t=0}^{T-1} \frac{h_i(\xi_t^i;\xi_t^{-i})}{T} = h_i(p) \tag{14}$$

$$\liminf_{T\to\infty} \sum_{t=0}^{T-1} \frac{h_i(\alpha_i(\xi^{-i}[t]);\xi_t^{-i})}{T} \leq h_i(p). \tag{15}$$

Because θ^i is $\bigotimes_{k\neq j}\theta^k$-computable for any $j \neq i$, ξ^j is θ^i-random for μ_{p^j} for each $j \neq i$. By Theorem 4.1, it follows that for any $j,k \neq i$, $\xi^j \otimes \xi^k$ is θ^i-random for $\mu_{p^j\otimes p^k}$. A simple induction argument shows that $\bigotimes_{j\neq i}\xi^j$ is θ^i-random for $\mu_{\otimes_{j\neq i}p^j}$ and that $\bigotimes_{j=1,\ldots,N}\xi^j$ is random for $\mu_{\otimes_{j=1}^N p^j}$. Then (14) follows from Theorem 1.2.

As for (15), let $\alpha_i \in \mathcal{A}_i$ be given. For each $y \in X_i$, let $r^y : X_{-i}^* \to \{0,1\}$ be the selection function for X_{-i} such that $r^y(\sigma) = 1$ if $\alpha_i(\sigma) = x$, and $r^y(\sigma) = 0$ otherwise. Notice that r^y is θ^i-computable. Because $\zeta \equiv \bigotimes_{j\neq i}\xi^j$ is θ^i-random for $\mu_{\otimes_{j\neq i}p^j}$, by Theorem 1.2,

$$\lim_{T\to\infty} \sum_{t=0}^{T-1} \frac{c_{x_{-i}}(\zeta_t^{r^y})}{T} = \prod_{j\neq i} p^j[x_j] \text{ for all } x_{-i} \in X_{-i}. \tag{16}$$

if ζ^{r^y} is an infinite sequence.

Define $L_y(T) = \#\{t \in \mathbb{N} : 0 \leq t \leq T-1,\ r^y(\zeta[t]) = 1\}$ and $\zeta^y = \zeta^{r^y}$. Let

$$\mathcal{E}^1 = \{y \in X_i : \lim_{T\to\infty} L_y(T) = \infty\} \text{ and } \mathcal{E}^2 = \{y \in X_i : \lim_{T\to\infty} L_y(T) < \infty\}.$$

For each $y \in \mathcal{E}^2$, let $B_y = \lim_{T\to\infty} L_y(T)$ and let $C_y = \sum_{t=0}^{B_y-1} h_i(y;\zeta_t^y)$. On the other hand, for any $y \in \mathcal{E}^1$, because ξ satisfies (16),

$$\lim_{T\to\infty} \sum_{t=0}^{T-1} \frac{h_i(y;\zeta_t^y)}{T} = \lim_{T\to\infty} \sum_{x_i\in X_{-i}} \sum_{t=0}^{T-1} \frac{c_{x_{-i}}(\zeta_t^y) h_i(y;x_{-i})}{T} = h_i(y;p^{-i}) \leq h_i(p).$$

I claim that for any $\varepsilon > 0$, there is some T' such that $T > T'$ implies that

$$\sum_{t=0}^{T-1} \frac{h_i(\alpha_i(\zeta[t]);\zeta_t)}{T} \leq h_i(p) + \varepsilon. \tag{17}$$

Fix some $\varepsilon > 0$. Let T_1 be so large that $T > T_1$ implies that, for all $y \in \mathcal{E}^1$, $\sum_{t=0}^{T-1} \frac{h_i(y;\zeta_t^y)}{T} \leq h_i(p) + \frac{\varepsilon}{2|X_1\times\ldots.\times X_N|}$, and, for all $y \in \mathcal{E}^2$, $\frac{C_y}{T} > -\frac{\varepsilon}{2|X_1\times\ldots\times X_N|}$. Let T' be so large that, for all $y \in \mathcal{E}_1$, $L_y(T') > T_1$ and $h_i(p) \sum_{y \in \mathcal{E}_1} \frac{L_y(T)}{T} \leq h_i(p) + \frac{\varepsilon}{2}$ for all $T > T'$. If $T > T'$, then

$$\sum_{t=0}^{T-1} \frac{h_i(\alpha_i(\zeta[t]); \zeta_t)}{T} = \sum_{y \in \mathcal{E}_1} \frac{L_y(T)}{T} \sum_{t=0}^{L_y(T)-1} \frac{h_i(y; \zeta_t^y)}{L_y(T)} + \sum_{y \in \mathcal{E}_2} \sum_{t=0}^{L_y(T)-1} \frac{h_i(y; \zeta_t^y)}{T}$$
$$\geq \sum_{y \in \mathcal{E}_1} \frac{L_y(T)}{T} \left(h_i(p) - \frac{\varepsilon}{2|X_1 \times .... \times X_N|} \right) - \sum_{y \in \mathcal{E}_2} \frac{\varepsilon}{2|X_1 \times .... \times X_N|} \geq h_i(p) + \varepsilon.$$

Notice that L_y is weakly increasing, and $L_y(T) \leq T$ for all T. Thus, $T > T'$ implies that $L_y(T) \geq L_y(T') > T_1$, and so $T > T_1$. This proves (17), which implies (15). □

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