

# Monetary Policy and Asset Prices: A Mechanism Design Approach\*

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## Abstract

We investigate the effects of monetary policy on asset prices in economies where assets are traded in over-the-counter (OTC) markets. The trading mechanism in pairwise meetings is designed to maximize social welfare taking as given the frictions in the environment (e.g., lack of commitment and limited enforcement) and monetary policy. We show that asset price "bubbles" emerge in a constrained-efficient monetary equilibrium only if liquidity is abundant and the first best is implementable. In contrast, if liquidity is scarce, assets are priced at their fundamental value in any constrained-efficient monetary equilibrium, in which case an increase in inflation has no effect on asset prices, but it reduces output and welfare. Finally, for low inflation rates the bursting of an asset price bubble can be an optimal response to a shock that reduces assets' resalability.

**JEL Classification:** D82, D83, E40, E50

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# 1 Introduction

How does monetary policy affect asset prices? A common view is that tightened monetary policy can contribute to the bursting of stock market bubbles. Shiller (2005, p.224) attributes the stock market downturn in the 30's to the tight monetary policy of the Federal Reserve, and the stock market crash in Japan in the early 90's to the decision of the Bank of Japan to raise its discount rate. Conversely, the rise of the housing market bubble in the U.S. is often linked to the policy of low interest rates implemented by the Federal Reserve following the 2001 recession. Among others, Shiller (2008, p.48) and Taylor (2007) notice that the period of low interest rates between mid-2003 and mid-2004 coincides with the period of most rapid home price increase.

Microfounded monetary models with their emphasis on liquidity have proved singularly useful to capture a transmission mechanism of monetary policy to asset prices. A case in point is the framework developed by Lagos and Wright (2005) that describes an economy where assets are traded and priced in competitive markets, as in Lucas (1978), and in over-the-counter (OTC) markets with bilateral meetings and bargaining, as in Duffie, Gârleanu, and Pedersen (2005). This model provides a theory of liquidity premia, or "bubbles," which can be used to establish how asset prices depend on the rate of return of currency.<sup>1</sup> However, its predictions regarding the relationship between monetary policy and asset prices seem at odds with the common wisdom: low nominal interest rates drive asset prices toward their "fundamental" value while high interest rates fuel bubbles. Moreover, from a methodological standpoint the standard approach is to assume that prices in OTC markets are set by mechanisms (e.g., Nash bargaining) that are socially inefficient, which makes it unclear whether asset pricing patterns generated by these models are due to the adoption of arbitrary (suboptimal) trading mechanisms, or if they should be viewed as robust features of monetary economies.

The objective of this paper is to study the channel through which monetary policy affects asset prices by taking seriously the observation that some assets are traded over the counter in bilateral meetings.<sup>2</sup> We will adopt the mechanism design approach promoted by Wallace (2001, 2010) according to which the trading mechanism in bilateral meetings in the OTC market is designed to maximize social welfare

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<sup>1</sup>Throughout the paper we will use the expressions "bubbles" and "liquidity premia" interchangeably. We define a bubble as a deviation of the asset price from its fundamental value, where the fundamental value is the sum of the asset's dividends discounted at the agent's rate of time preference. Alternatively, the fundamental value is the price that would prevail in an economy with no credit friction where assets play no liquidity role.

<sup>2</sup>See Duffie (2012, ch.1) for a description of the institutional setting of over-the-counter markets and some key issues associated with market opaqueness.

given the various frictions that plague the economic environment (e.g., lack of commitment, limited enforcement, and imperfect record-keeping). Specifically, the trading mechanism cannot command trades that are not incentive feasible, and it leaves no room for any mutually beneficial renegotiation. As a result, the asset pricing patterns that emerge out of this mechanism will reflect the elementary frictions that create the need for liquidity. Moreover, the mechanism design approach allows us to distinguish components of asset prices that are essential to achieve good allocations from components that are not essential—they could be eliminated without affecting welfare. One can think of the former as liquidity premia and the latter as rational bubbles. In a nutshell, our model predicts a relationship between monetary policy and asset prices that is broadly consistent with the common wisdom: (i) low costs of holding fiat money make the emergence of asset price bubbles possible; (ii) tight monetary policies eliminate bubbles as long as the holding cost of fiat money is not large enough to threaten the monetary equilibrium.

We will consider an economy composed of two assets: fiat money and a Lucas tree that yields a positive dividend flow in every period. Both assets can serve as media of exchange (e.g., means of payment or collateral) to finance random consumption opportunities in the OTC market.<sup>3</sup> The presence of fiat money in the model will allow us to study how the growth rate of money supply (that is not part of the mechanism design problem) affects asset prices. In the benchmark version of the model Lucas trees are long lived and the supply of fiat money is constant. We establish that fiat money plays no essential role: any constrained-efficient allocation that can be implemented with a valued fiat money can also be implemented with an asset price bubble attached to Lucas trees. A bubble emerges if the supply of the asset valued at its fundamental price is too low to finance the first-best trades in the OTC market. If the asset supply is in some intermediate range, then an asset price bubble is essential to implement the first-best level of output.

We generalize the model by considering the case where Lucas trees are short-lived and depreciate over time and the money supply grows (or shrinks) at a constant rate. Our model has the following implications for monetary policy summarized in Figure 1. First, the first-best allocation is implementable provided that the inflation rate is below a threshold (greater than the Friedman rule), and this threshold increases with the supply of Lucas trees. For such low inflation rates reducing the cost of holding fiat

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<sup>3</sup>As shown in Li, Rocheteau, and Weill (2011) this model can readily be reinterpreted as a model of an OTC market for bilateral risk sharing, such as the market of interest-rate swaps, where agents trade insurance services and use assets as collateral.

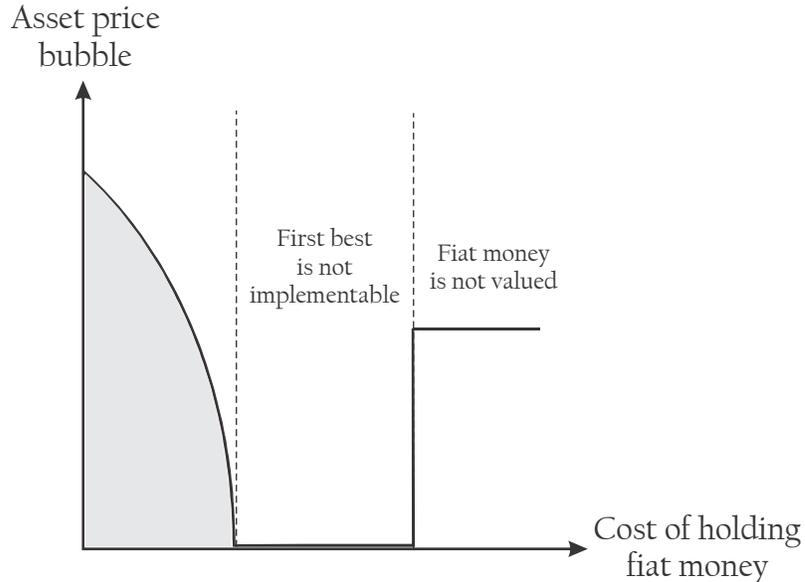


Figure 1: Monetary policy and asset prices: An overview.

money raises the maximum size of the bubble on Lucas trees that is consistent with an optimal allocation. (See shaded area in Figure 1.) So the model captures the view that a policy of cheap liquidity can fuel asset price bubbles. However, when inflation rate is in this range, such bubbles are inessential in the sense that the liquidity premium can also be attached to the fiat money (and the assets are traded at the fundamental value) to implement the optimal allocation.

Second, when the inflation rate is in some intermediate range, so that the first best is no longer achievable, then Lucas trees are priced at their fundamental value. (See the region between dotted lines in Figure 1.) This finding is consistent with the view that a tightened monetary policy can trigger the bursting of an asset price bubble. In this case an increase in inflation has no effect on asset prices but it reduces output. Moreover, Lucas trees have a higher rate of return than fiat money, an example of rate-of-return dominance. Third, if inflation is sufficiently large, then it is socially optimal to substitute a valued fiat money with a bubble on productive assets, i.e., a constrained-efficient monetary equilibrium no longer exists.

Finally, following Kiyotaki and Moore (2005), we introduce a resaleability constraint specifying that an agent can only transfer, or collateralize, a fraction of his Lucas trees in a bilateral meeting in the OTC market.<sup>4</sup> We use this formalization to study the optimal response of the economy to a shock that

<sup>4</sup>For related formalizations, see also Lagos (2010) and Lester, Postlewaite, and Wright (2012). In the Appendix we

reduces the resalability of Lucas trees. If the inflation rate is not too large, a lower resalability of trees results in the reduction or the elimination of asset price bubbles. Moreover, such a shock reduces the upper bound for the cost of holding fiat money that is consistent with a first-best allocation. This result suggests that, to maintain the first-best allocation, a shock that reduces the resaliability of real assets must be accompanied by a more accommodative monetary policy.

## 1.1 Related Literature

This paper is part of a recent literature, surveyed in Nosal and Rocheteau (2011) and Williamson and Wright (2010), that explains the liquidity of assets by their role as means of payment or collateral to overcome frictions in monetary economies. The first papers to include Lucas trees in such models are Lagos (2010, 2011) and Geromichalos, Licari, and Suarez-Lledo (2007). Resalability constraints are introduced by Kiyotaki and Moore (2005), Lagos (2010), and Lester, Postlewaite, and Wright (2012). This class of models has been used to explain the equity premium and risk-free rate puzzles, (Lagos, 2010, 2011), the excess volatility puzzle (Ravikumar and Shao, 2010), the dynamics of asset price bubbles (Kocherlakota, 2009; Rocheteau and Wright, 2010), and the effects of monetary policy on asset prices (Geromichalos, Licari, and Suarez-Lledo, 2007). The results from this literature differ significantly from ours. For instance, it is typically found that liquidity premia emerge only if equilibrium output is inefficiently low, and policies that drive the cost of holding fiat money to zero (Friedman rule) eliminate such liquidity premia. Moreover, in the absence of resalability constraints assets and money have the same rate of return. In contrast, under an optimal trading mechanism assets are priced at their fundamental value in any monetary equilibrium where output is low relative to the first best, and liquidity premia occur for low costs of holding fiat money. Moreover, fiat money and assets do not need to have the same rate of return.

The description of the asset market as decentralized with pairwise meetings is related to the finance literature on over-the-counter markets pioneered by Duffie, Gârleanu, and Pedersen (2005) and extended by Weill (2007) and Lagos and Rocheteau (2009). In contrast to our approach, prices are determined through Nash bargaining and agents have unlimited access to credit in pairwise meetings in the OTC market. This literature is surveyed in Rocheteau and Weill (2011) and Duffie (2012).

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endogenize the resalability of Lucas trees by assuming that agents must invest in a costly technology to authenticate assets. Similar informational frictions are formalized explicitly in Li, Rocheteau and Weill (2011), Rocheteau (2011), Hu (2013), and Lester, Postlewaite and Wright (2012).

The use of mechanism design in monetary theory has been advocated by Wallace (2010). It has been applied to the Lagos-Wright model by Hu, Kennan, and Wallace (2009) to show that the Friedman rule is not necessary to achieve good allocations, and in some cases it is not feasible. Hu and Rocheteau (2013) applies a similar approach to a model with fiat money and produced capital in order to show that the coexistence of money and higher-return assets is property of optimal allocations in monetary economies.

The rest of the paper is organized as follows. Section 2 describes the economic environment. Section 3 characterizes the set of implementable allocations. In Section 4 we determine constrained-efficient outcomes with and without fiat money. Section 5 introduces money growth to study the relationship between inflation and asset prices. We investigate the effects of resalability constraints in Section 6.

## 2 The environment

Time is discrete, preferences are additively separable over dates (and stages), and there is a nonatomic unit measure of agents divided evenly into a set of buyers,  $\mathbb{B}$ , and a set of sellers,  $\mathbb{S}$ . Each date has two stages: first pairwise meetings (OTC market) and then a centralized meeting. The first stage will be referred to as DM (decentralized market) while the second stage will be referred to as CM (centralized market). There is a single good at each stage. The CM good will be taken as the numéraire. The labels “buyer” and “seller” refer to agents’ roles in the DM: only the sellers can produce the DM good and only the buyers desire DM goods.<sup>5</sup> In the DM a fraction  $\sigma \in (0, 1]$  of buyers meet with sellers in pairs.

Agents maximize expected discounted utility with discount factor  $\beta = \frac{1}{1+r} \in (0, 1)$ . The DM utility of a seller who produces  $y \in \mathbb{R}_+$  is  $-v(y)$ , while that of a buyer who consumes  $y$  is  $u(y)$ , where  $v(0) = u(0) = 0$ ,  $v$  and  $u$  are strictly increasing and differentiable with  $v$  convex and  $u$  strictly concave, and  $u'(0) > v'(0)$ . Moreover, there exists  $\tilde{y} > 0$  such that  $v(\tilde{y}) = u(\tilde{y})$ . We denote  $y^* = \arg \max [u(y) - v(y)] > 0$  the quantity that maximizes a match surplus. The utility of consuming  $z \in \mathbb{R}$  units of the numéraire good is  $z$ , where  $z < 0$  is interpreted as production.

In addition to those perishable goods, there are two types of assets, a Lucas (1978) tree and fiat money. The supply of Lucas trees per buyer is  $A$ , and the supply of fiat money per buyer is normalized to 1. Assets are perfectly divisible. Fiat money is infinitely durable and intrinsically useless, i.e., it

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<sup>5</sup>We adopt the version of the model where an agent’s type, buyer or seller in the DM, is permanent because it simplifies the presentation of the model without affecting the main insights.

generates no flow of output and no direct utility. In contrast, each unit of the Lucas tree generates one unit of numéraire good.<sup>6</sup> After dividends have been paid, but before the CM opens, each unit of the tree depreciates at rate  $\delta \in [0, 1]$ . In the CM buyers receive a flow  $\delta A$  of new trees, which can be traded immediately. Trees of different vintages are indistinguishable.

People cannot commit to future actions, and there is no monitoring (histories are private information)—assumptions that serve to make some assets essential as means of payment in bilateral matches. While it is not crucial for our results, we assume that people can hide assets but they cannot overstate their asset holdings. With no loss in generality we will restrict sellers to hold no asset across periods.

### 3 Implementation

We study equilibrium outcomes that can be implemented by mechanism designer proposals. A *proposal* consists of three objects: (i) a function in the bilateral matches,  $o : \mathbb{R}_{2+} \times \mathbb{N}_0 \rightarrow \mathbb{R}_{3+}$ , that maps the buyer's announced asset holdings of trees and money,  $(a, m)$ , and time,  $t$ , into a proposed allocation,  $(y, \tau_a, \tau_m) \in [0, \tilde{y}] \times [0, a] \times [0, m]$ , where  $y$  is the DM output produced by the seller and consumed by the buyer,  $\tau_a$  is the transfer of trees, and  $\tau_m$  is the transfer of money; (ii) an initial distribution of money,  $\mu$ ; (iii) a sequence of prices for money in terms of the numéraire good,  $\{\phi_t\}_{t=0}^\infty$ , and a sequence of prices for the Lucas tree in terms of the numéraire good,  $\{q_t\}_{t=0}^\infty$ , in the second-stage CM.

The trading mechanism in the DM is as follows: the buyer first announces his asset holdings; then both the buyer and the seller simultaneously respond with *yes* or *no*: if both respond with *yes*, then the proposed trade is carried out; otherwise, there is no trade. This ensures that trades are individually rational. We also require any proposed trades to be in the pairwise core.<sup>7</sup> We assume that agents in the CM trade competitively against the proposed prices. This is consistent with the pairwise core requirement in the DM due to the equivalence between the core and competitive equilibria.

We denote  $s_b$  the strategy of buyer  $b \in \mathbb{B}$ , which consists of three components for any given trading history  $h^t$  at the beginning of period  $t$ :  $s_b^{h^t,1}(a, m) \in [0, a] \times [0, m] \times \{yes, no\}$  that maps his asset holdings to his announcements and first-stage response in the DM; (ii)  $s_b^{h^t,2}(a, m, y, \tau_a, \tau_m) \in [0, a] \times [0, m]$  that

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<sup>6</sup>It would be equivalent to assume that the dividend of a tree is the realization of a stochastic process provided that agents do not have information about the realization of the dividend prior to the CM.

<sup>7</sup>The pairwise core requirement can be implemented directly with a trading mechanism that adds a renegotiation stage as in Hu, Kennan, and Wallace (2009). The renegotiation stage will work as follows. An agent will be chosen at random to make an alternative offer to the one made by the mechanism. The other agent will then have the opportunity to choose among the two offers.

maps the buyer's asset holdings upon entering CM and consumption in DM to his final asset holdings after the CM. Similarly, we denote  $s_s$  the strategy of seller  $s \in \mathbb{S}$ , which consists of three components for any given trading history  $h^t$  at the beginning of period  $t$ : (i)  $s_s^{h^t,1}(a, m) \in [0, a] \times [0, m] \times \{yes, no\}$  that maps the buyer's announcement to his stage-1 response in the DM; (ii)  $s_s^{h^t,2}(a, m, y, \tau_a, \tau_m) \in [0, a] \times [0, m]$  that maps the seller's asset holdings upon entering CM and production in the DM to his final asset holdings after the CM.

**Definition 1** *An equilibrium is a list,  $\langle (s_b : b \in \mathbb{B}), (s_s : s \in \mathbb{S}), o, \mu, \{q_t, \phi_t\}_{t=0}^\infty \rangle$ , composed of one strategy for each agent in  $\mathbb{B} \cup \mathbb{S}$  and the planner proposals  $(o, \mu, \{q_t, \phi_t\}_{t=0}^\infty)$  such that: (i) Each strategy is sequentially rational given other players' strategies and asset prices; (ii) The centralized meeting clears at every date.*

Throughout the paper we restrict our attention to equilibria that involve stationary proposals and that use symmetric and stationary strategies in which the buyer is always truthful in announcements and both the buyer and the seller respond with *yes* in all DM meetings, the initial distribution of money is uniform across buyers, and money and asset prices are constant over time. We call such equilibria *simple* equilibria. In a simple equilibrium, the equilibrium outcome is characterized by a list  $(\bar{y}, \bar{\tau}_a, \bar{\tau}_m, q, \phi)$ , where  $(\bar{y}, \bar{\tau}_a, \bar{\tau}_m)$  is the trade in all matches in the DM,  $\phi$  is the price for money in all periods, and  $q$  is the price for Lucas trees in all periods. Such an equilibrium outcome  $(\bar{y}, \bar{\tau}_a, \bar{\tau}_m, q, \phi)$  is *implementable* if it is the equilibrium outcome for a simple equilibrium associated with a proposal  $o$ .

We begin with the value functions along the equilibrium path in a simple equilibrium for a given proposal  $(o, q, \phi)$ . Here we denote the components of  $o$  by  $o(a, m) = (y(a, m), \tau_a(a, m), \tau_m(a, m))$ . Let  $V^b(a, m)$  and  $W^b(a, m)$  denote the continuation value for a buyer with asset holding  $(a, m)$  upon entering the DM and CM, respectively; similarly, let  $V^s$  denote the continuation value for a seller upon entering the DM and let  $W^s(a, m)$  denote the continuation value for a seller with asset holding  $(a, m)$  upon entering the CM.

**Lemma 1** *Given the proposed equilibrium outcome  $(\bar{y}, \bar{\tau}_a, \bar{\tau}_m, q, \phi)$ , the value functions that are consis-*

tent with a simple equilibrium are:

$$V^b(a, m) = \sigma \{u[y(a, m)] - [(1 - \delta)q + 1] \tau_a(a, m) - \phi \tau_m(a, m)\} + W^b(a, m); \quad (1)$$

$$W^b(a, m) = \phi m + [(1 - \delta)q + 1]a + W^b(0, 0); \quad (2)$$

$$W^b(0, 0) = -\phi - (1 - \delta)qA + \beta \frac{\sigma [u(\bar{y}) - \bar{\tau}_a [(1 - \delta)q + 1] - \bar{\tau}_m \phi] + A}{1 - \beta}; \quad (3)$$

$$V^s = \frac{\sigma \{-v(\bar{y}) + [(1 - \delta)q + 1] \bar{\tau}_a + \bar{\tau}_m \phi\}}{1 - \beta}; \quad (4)$$

$$W^s(a, m) = \phi m + [(1 - \delta)q + 1]a + \beta V^s. \quad (5)$$

All proofs are in Appendix A. These expressions for value functions follow from standard arguments. First notice that, because we restrict the sellers not to carry assets across periods,  $V^s$  does not depend on the seller's asset holdings. This is with no loss of generality: since sellers do not consume in the DM, holding assets across periods is costly because of discounting. For the same reason, when entering the CM with asset holding  $(a, m)$ , the optimal action is to sell all the assets, which results in consumption of the CM good equal to  $\phi m + [(1 - \delta)q + 1]a$ . Notice that  $a$  units of trees pay  $a$  units of CM good as dividend and become  $(1 - \delta)a$  units after depreciation.

Now we turn to the buyer's value functions. Because of quasi-linearity of the CM preference and because of the competitive trading protocol in the CM, standard arguments show that

$$W^b(a, m) = \phi m + [(1 - \delta)q + 1]a + W^b(0, 0),$$

that is, the portfolio choice in the CM is independent of the buyer's asset holdings when entering the CM. When calculating  $W^b(0, 0)$ , we use the budget constraint according to which the net consumption of the numéraire good is  $z = \delta qA - \phi \hat{m} - q\hat{a}$ , where the first term is a lump-sum endowment of Lucas trees to replace the depreciating trees and the last two terms correspond to the value of the buyer's end-of-period portfolio. On equilibrium path,  $(\hat{a}, \hat{m}) = (A, 1)$ , and, by definition,  $o(A, 1) = (\bar{y}, \bar{\tau}_a, \bar{\tau}_m)$ .<sup>8</sup>

Here we give another requirement on the proposal. In accordance with the notion of coalition-proof implementability of HKW we restricted the mechanism in the DM to propose trades in the pairwise core. Given a buyer's portfolio,  $(a, m)$ , and asset prices,  $(q, \phi)$ , and using the linearity of the value

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<sup>8</sup>The Belleman equations listed are valid for any trading protocol in the DM, including Nash bargaining, proportional bargaining, and the ultimatum game.

functions in the CM, the set of coalition-proof allocations is

$$\mathcal{C}(a, m, q, \phi) = \arg \max_{y, \tau_a, \tau_m} \{u(y) - \tau_a [(1 - \delta)q + 1] - \tau_m \phi\} \quad (6)$$

$$\text{s.t. } (\tau_a, \tau_m) \in [0, a] \times [0, m] \quad (7)$$

$$-v(y) + \tau_a [(1 - \delta)q + 1] + \tau_m \phi \geq U^s, \quad (8)$$

for some  $U^s \in [0, U_{\max}^s(a, m, q, \phi)]$ , where

$$U_{\max}^s(a, m, q, \phi) = u(y^*) - v(y^*) \text{ if } u(y^*) \leq a [(1 - \delta)q + 1] + m\phi; \quad (9)$$

$$U_{\max}^s(a, m, q, \phi) = a [(1 - \delta)q + 1] + m\phi - v[u^{-1}[a((1 - \delta)q + 1) + m\phi]] \text{ otherwise.}$$

Now we are ready to present a characterization of implementable outcomes.

**Proposition 1** *An equilibrium outcome,  $(\bar{y}, \bar{\tau}_a, \bar{\tau}_m, q, \phi)$ , is implementable if and only if it satisfies*

$$-r\phi - [(r + \delta)q - 1]A + \sigma \{u(\bar{y}) - [(1 - \delta)q + 1] \bar{\tau}_a - \bar{\tau}_m \phi\} \geq 0 \quad (10)$$

$$-v(\bar{y}) + \bar{\tau}_m \phi + [(1 - \delta)q + 1] \bar{\tau}_a \geq 0 \quad (11)$$

$$q - \frac{1}{r + \delta} \geq 0 \quad (12)$$

and  $(\bar{y}, \bar{\tau}_a, \bar{\tau}_m) \in \mathcal{C}(A, 1, q, \phi)$ .

*Proposals in the DM consistent with this outcome are as follows. If  $a \geq A$  and  $m \geq 1$ , then*

$$o(a, m) = \arg \max_{y, \tau_m \leq m, \tau_a \leq a} \{-v(y) + \phi \tau_m + [(1 - \delta)q + 1] \tau_a\} \quad (13)$$

$$\text{s.t. } u(y) - \phi \tau_m - [(1 - \delta)q + 1] \tau_a \geq u(\bar{y}) - \bar{\tau}_m \phi - [(1 - \delta)q + 1] \bar{\tau}_a.$$

*If  $a < A$  or  $m < 1$ , then*

$$o(a, m) = \arg \max_{y, \tau_m \leq m, \tau_a \leq a} \{-v(y) + \phi \tau_m + [(1 - \delta)q + 1] \tau_a\} \quad (14)$$

$$\text{s.t. } u(y) - \phi \tau_m - [(1 - \delta)q + 1] \tau_a \geq 0.$$

Inequalities (10) and (11) provide two individual-rationality (IR) constraints, one for buyers in the CM and one for sellers in the DM. Inequality (10) gives a necessary condition for buyers to hold the equilibrium portfolio  $(A, 1)$ : their expected surplus in the DM,  $\sigma \{u(\bar{y}) - [(1 - \delta)q + 1] \bar{\tau}_a - \bar{\tau}_m \phi\}$ , net of the cost of holding the assets,  $r\phi + [(r + \delta)q - 1]A$ , has to be nonnegative. The cost of holding an asset is equal to the difference between the cost of investing into the asset, as measured by the sum of

the rate of time preference and the rate of depreciation of the asset times the price of the asset, and the expected dividend of the asset. In the case of fiat money the depreciation rate is 0 and the dividend is 0. Inequality (11) states that sellers are willing to go along with the proposed equilibrium trade only if their surplus in the DM is nonnegative. Finally, (12) indicates that in any equilibrium the price of a Lucas tree cannot be less than its fundamental value,  $q^* = (r + \delta)^{-1}$ , as measured by the discounted sum of its dividends, for otherwise the buyer would like to have unbounded holdings of the Lucas trees and this would in turn raise the price above  $q^*$ .

For any equilibrium outcome that satisfies (10)-(12) and the pairwise core requirement, we construct a planner proposal in (13)-(14) that implements it. According to (13), if the buyer holds at least  $A$  units of trees and at least 1 unit of fiat money, then the mechanism selects the pairwise Pareto-efficient allocation that gives the buyer the same surplus as the one he would obtain under the trade  $(\bar{y}, \bar{\tau}_a, \bar{\tau}_m)$ . As a consequence, buyers have no incentives to accumulate more than  $A$  units of trees and more than one unit of money. According to (14), if the buyer holds less than  $A$  units of trees or less than one unit of fiat money, then the mechanism chooses the allocation that maximizes the seller's surplus subject to the buyer being indifferent between trading and not trading. This guarantees that the buyer has no incentive to bring less than the equilibrium portfolio,  $(A, 1)$ .

## 4 Essential bubbles

Society's welfare is measured by the discounted sum of buyers' and sellers' utility flows, i.e.,

$$\mathcal{W}(y) = \sum_{t=1}^{\infty} \frac{\beta^t}{2} \{ \sigma [u(y) - v(y)] + A \}. \quad (15)$$

In each period, a measure  $\sigma/2$  of matches are formed. The total surplus of each match is  $u(y) - v(y)$ . From (15) our welfare criterion only concerns the level of trade,  $y$ , in the pairwise meetings. In the following we say that the level of DM output,  $y$ , is implementable if there exists an implementable equilibrium outcome  $(y, \tau_a, \tau_m, q, \phi)$ . The first-best allocation is such that  $y = y^*$ .

**Definition 2** *A constrained-efficient outcome is a list  $(y, \tau_a, \tau_m, q, \phi)$  that maximizes  $u(y) - v(y)$  among all implementable equilibrium outcomes.*

We can simplify our search for constrained-efficient outcomes by noticing the following.

**Lemma 2** *There exist some  $\bar{\tau}_a \leq A$  and  $\bar{\tau}_m \leq 1$  such that  $(\bar{y}, \bar{\tau}_a, \bar{\tau}_m, q, \phi)$  is a constrained-efficient outcome if and only if  $(\bar{y}, q, \phi)$  solves  $\max_{y \leq y^*} y$  subject to  $q = q^* + \ell$ ,  $\ell \geq 0$ ,  $\phi \geq 0$ ,*

$$-r\phi - (r + \delta)\ell A + \sigma [u(y) - v(y)] \geq 0 \quad (16)$$

$$\phi + (1 - \delta)\ell A + \frac{1 + r}{\delta + r} A - v(y) \geq 0. \quad (17)$$

We say that  $(y, q, \phi)$  is a constrained efficient outcome if it solves the maximization problem in the lemma. The term  $\ell$  is the difference between the price of the Lucas trees and their fundamental value, which can be interpreted as a *liquidity premium* or a *bubble*. Inequalities (16) and (17) are derived from (11) and (10), but weaker. The left side of (16) is the expected surplus of a match net of the cost of holding assets, and the left side of (17) is the value of aggregate wealth net of disutility for production in the DM. These constraints are satisfied if by allocating all the match surplus to the buyer, the buyer would be willing to participate in the CM and to hold sufficiently large quantity of assets, and there are enough assets to compensate the seller for his disutility of production. Lemma 2 shows that, to look for constrained-efficient outcomes, it is sufficient to search among outcomes satisfying those weaker conditions, as well as the two conditions on asset prices. In particular, it implies that the pairwise core requirement is not binding as far as constrained-efficient outcomes are concerned.

Now we solve the optimization problem in Definition 2. We begin with the case where the Lucas trees are the only assets that can be used as means of payment in the DM, i.e., we impose  $\phi = 0$ . If the first-best is achievable without money, then money is not essential in the sense of Wallace (2001). To characterize constrained-efficient outcomes it is useful to define the following thresholds for the supply of assets,

$$A_1 = \frac{r + \delta}{1 + r} v(y^*) \quad (18)$$

$$A_2 = \frac{(r + \delta)v(y^*) - (1 - \delta)\sigma [u(y^*) - v(y^*)]}{1 + r}. \quad (19)$$

It is straightforward to verify that  $A_1 > A_2$  if and only if  $\delta < 1$ . Throughout the paper we assume that

$$\frac{\sigma [u(y^*) - v(y^*)]}{v(y^*)} < r \quad (20)$$

to make things interesting. Inequality (20) implies that  $A_2 > 0$ . In the following proposition we say that  $(y, q)$  is a constrained efficient equilibrium outcome if it solves the welfare-maximization problem with the additional constraint  $\phi = 0$ .

**Proposition 2 (Asset prices in a nonmonetary economy.)** Suppose that  $\phi = 0$ .

(i)  $(y^*, q^*)$  is a constrained-efficient outcome if and only if  $A \geq A_1$ .

(ii) Suppose that  $A \in [A_2, A_1)$ . The outcome,  $(y, q)$ , is constrained efficient if and only if  $y = y^*$  and  $q = q^* + \ell$  with

$$[(1+r)(A_1 - A)]/[(r+\delta)(1-\delta)] \leq \ell \leq \sigma[u(y^*) - v(y^*)]/(r+\delta)A \equiv \bar{\ell}(A). \quad (21)$$

(iii) Suppose that  $A < A_2$ . The constrained-efficient outcome is  $(\bar{y}, q)$  with  $\bar{y} < y^*$  and  $q = q^* + \ell$  where  $\ell$  solves

$$\sigma[u(\bar{y}) - v(\bar{y})]/(r+\delta) = v(\bar{y})/(1-\delta) - (1+r)A/[(r+\delta)(1-\delta)] = A\ell. \quad (22)$$

The threshold  $A_1$  in part (i) matches exactly the liquidity needs of the economy, as captured by the amount of wealth that is necessary to compensate sellers for the production of the first-best level of output,  $v(y^*)$ , when the asset is priced at its fundamental value  $q = q^*$ . Notice that with  $q = q^*$  and  $\phi = 0$ , the buyer's IR constraint, (16), is never binding. This result is similar to the one in Geromichalos, Licari, and Suarez-Lledo (2007) or Lagos (2011) when buyers set the terms of trade unilaterally. When the asset supply is abundant ( $A \geq A_1$ ), no liquidity premium (or a bubble) is essential (to implement  $y^*$ ).

If  $A < A_1$  but  $A \geq A_2$  the first-best allocation cannot be implemented with  $q = q^*$  but it is implementable only with a liquidity premium (a bubble) on the asset prices. In order to understand the threshold  $A_2$ , consider the largest liquidity premium  $A\ell$  that is consistent with the buyer's IR constraint in the CM, (16), is

$$A\ell = \frac{\sigma}{r+\delta} [u(y^*) - v(y^*)]. \quad (23)$$

This also gives the upper bound in (21). The buyer is willing to carry the asset with bubble  $\ell$  because the mechanism can punish buyers who reduce their asset holdings by taking away the match surplus,  $u(y^*) - v(y^*)$  whenever a match occurs. From the seller's IR constraint in the DM, (17), the quantity of assets  $A$  needed by the buyer to compensate the seller for the production of the first-best level of output is determined by  $[(1-\delta)(q^* + \ell) + 1]A = v(y^*)$ , which can be reexpressed as

$$A = \frac{r+\delta}{1+r}v(y^*) - \frac{(r+\delta)(1-\delta)}{1+r}A\ell. \quad (24)$$

Substituting the size of the bubble given by (23) into (24), one finds the expression for  $A_2$  given by (19). Part (ii) then shows that the supply of assets can be lower than  $A_1$  and still allow for the

implementation of the first best provided that the asset price exhibits a bubble. The lower bound in (21) is obtained by considering the minimum bubble that is necessary to compensate seller's disutility in the DM. When  $A \in [A_2, A_1)$ , a positive liquidity premium is sustainable to implement the first-best allocation under an optimal mechanism. This finding that bubbles can coexist with first-best allocations contrasts sharply with Geromichalos, Licari, and Suarez-Lledo (2007) and Lagos (2011) where liquidity premia emerge only when the level of output is inefficiently low.

Finally, if  $A < A_2$ , then a first-best allocation is not implementable. Both the buyer's participation constraint in the CM and the seller's participation constraint in the DM bind. As a consequence, the output traded in bilateral matches is inefficiently low and the asset price in the CM exhibits a liquidity premium. From (22) an increase in the supply of the asset increases the aggregate liquidity premium,  $A\ell$ , and the asset price is increasing with the frequency of trades in the DM,  $\sigma$ .

Now we introduce a constant stock of fiat money and study whether it increases welfare when  $y = y^*$  is not implementable with the real asset alone. The following threshold for the supply of assets will turn out to be crucial to answer this question:

$$A_3 = (r + \delta) \{rv(y^*) - \sigma [u(y^*) - v(y^*)]\} / [r(1 + r)]. \quad (25)$$

This threshold comes from the following exercise. Let  $\bar{\phi}$  be the highest value that money can take when the Lucas trees are priced at their fundamental value,  $q = q^*$ , and agents trade the first-best level of output,  $\bar{y} = y^*$ . That is,  $\bar{\phi}$  is derived from (16) at equality,

$$\bar{\phi} = \sigma [u(y^*) - v(y^*)] / r. \quad (26)$$

The maximum value for fiat money is the discounted sum of all match surpluses at a first-best allocation. From (17), for a given value of money the minimum quantity of assets that is required to compensate a seller for the production of  $y^*$  is given by  $A = (r + \delta) [v(y^*) - \phi] / (1 + r)$ . Substituting  $\phi$  by the expression given by (26) gives the threshold  $A_3$ . It is straightforward to verify that  $A_3 \leq A_2$  and  $A_3 = A_2$  if and only if  $\delta = 0$ .

**Proposition 3 (*Asset prices and money.*)** Consider an economy with fiat money. Assume  $A < A_1$ .

(i) Suppose that  $A \in [A_2, A_1)$ . The outcome,  $(y, q, \phi)$ , is constrained efficient only if  $y = y^*$ ,  $\ell + \phi > 0$ , and  $\ell \leq \bar{\ell}(A)$ .

(ii) Suppose that  $A \in [A_3, A_2)$ . The outcome,  $(y, q, \phi)$ , is constrained efficient only if  $y = y^*$ ,  $\phi > 0$ ,

and  $\ell \leq \left\{ \sigma[u(y^*) - v(y^*)] - r \left[ v(y^*) - \frac{1+r}{\delta+r} A \right] \right\} / \delta(1+r)A$ .

(iii) Suppose that  $A < A_3$  and  $\delta > 0$ . The unique constrained-efficient outcome is  $(\bar{y}, q^*, \phi)$  where  $\bar{y} < y^*$  and  $\phi$  solves

$$\sigma[u(\bar{y}) - v(\bar{y})] = r \left[ v(\bar{y}) - \frac{1+r}{r+\delta} A \right] = r\phi. \quad (27)$$

(iv) Suppose that  $A < A_3$  and  $\delta = 0$ . Any constrained-efficient outcome,  $(\bar{y}, q, \phi)$ , is such that  $\bar{y} < y^*$  solves (27) and  $q = q^* + \ell$  where  $(\ell, \phi)$  solves  $\phi + \ell A = v(\bar{y}) - \frac{1+r}{r} A$ .

When the first-best is implementable, that is, when  $A \geq A_3$ , there is a continuum of asset prices  $(\ell, \phi)$  that are consistent with the constrained efficient outcome. Indeed, in Proposition 3 (i) and (ii), we report an upper bound for liquidity premium  $\ell$  on the Lucas trees that is consistent with the constrained efficient outcome. For any  $\ell$  below the upper bound, there is a range of prices for money, determined by (16) and (17). Here we emphasize that when  $A \in [A_2, A_1)$ , a positive liquidity premium is necessary but can be placed on either money or the Lucas trees; on the other hand, when  $A \in [A_3, A_2)$ , money has to carry a positive value to implement the constrained efficient outcome, although part of the premium can be placed on the trees as well.

Proposition 3 shows that if the Lucas trees are short-lived, i.e.,  $\delta > 0$ , then the introduction of fiat money is necessary to implement the first-best allocation for a range of parameter values ( $A \in [A_3, A_2)$ ) for which the Lucas trees alone cannot achieve. Moreover, when  $\delta > 0$ , even when  $y = y^*$  is not implementable, the presence of fiat money increases the level of DM output. To see this, it is easy to check that the solution of  $y$  to (22) is smaller than that to (27) since  $r < \frac{r+\delta}{1-\delta}$  if  $\delta > 0$ . Therefore, fiat money plays an essential role and it is socially optimal to price Lucas trees at their fundamental value.

The core argument for this result is the comparison of the effective cost of holding trees relative to that of holding money. The effective cost of holding the bubble attached to the tree to finance DM consumption is  $\frac{r+\delta}{1-\delta}$ : from the buyer's IR constraint, (16), the cost of holding a bubble of size  $\ell$  is  $(r+\delta)\ell$ ; from the seller's IR constraint, (17), this bubble is only worth  $(1-\delta)\ell$  as the trees depreciate at rate  $\delta$ . In contrast, the cost of holding fiat money is  $r$ . Therefore,  $\delta > 0$  implies that it is more efficient to use the fiat money instead of the Lucas trees to carry a bubble. Put it differently, the asset with the highest durability is better able to serve as a medium of exchange since it reduces the cost of holding liquidity, which relaxes the buyer's IR constraint in the CM. This finding is consistent with the idea that the physical properties of assets matter for their moneyness (Wallace, 1998). In equilibrium, the net rate of return of fiat money is 0 while the rate of return of the Lucas trees is  $r > 0$ . So rate-of-return

dominance is part of a constrained efficient outcome.<sup>9</sup>

Finally, if Lucas trees are infinitely-lived,  $\delta = 0$ , as in Geromichalos, Licari, and Suarez-Lledo (2007) and Lagos (2011), then the presence of fiat money does not raise social welfare. Any allocation that can be implemented with fiat money and the Lucas tree can be implemented with the Lucas tree alone. In other words, fiat money is not essential as the Lucas trees can serve the role of media-of-exchange equally well.

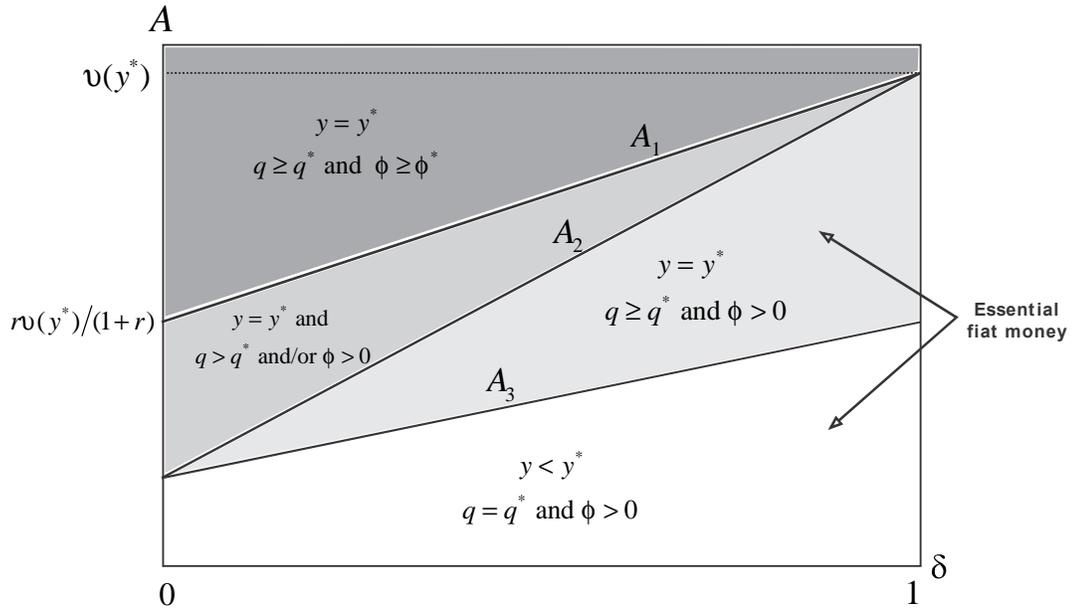


Figure 2: Essential bubbles and fiat money

Figure 2 illustrates graphically Propositions 2 and 3. It indicates the level of DM output and asset prices under constrained-efficient outcomes for different values of the supply of Lucas trees and the depreciation rate of trees. In the dark-grey area the first-best level of output can be implemented when the asset is priced at its fundamental value, even without the introduction of fiat money. This regime requires the asset to be sufficiently abundant (larger than  $A_1$ ). The larger the depreciation rate of trees, the larger the supply of assets required to implement the first best. In the medium grey area, when the

<sup>9</sup>This rate-of-return dominance result depends on our assumption that replacement trees are identical to original ones. This result will not hold if agents can recognize trees with different ages; see the concluding remarks for more details. Zhu and Wallace (2007) are also able to generate rate-of-return dominance in a related model with multiple assets and bilateral matches. However, their mechanism is suboptimal in that the buyer does not receive the full surplus of a match in equilibrium. Rocheteau (2011) studies a model with money and (produced) capital and shows that whenever money is essential rate-of-return dominance is a property of any equilibrium under an optimal mechanism.

asset supply is between the two thresholds  $A_1$  and  $A_2$ , the first-best level of output is implementable but it requires either the price of trees to carry a bubble or fiat money to be valued. In the light-grey area, when the supply of Lucas trees is between  $A_2$  and  $A_3$ , fiat money is essential in the sense that it must be valued to implement the first-best allocation. Notice that this region is non-degenerate provided that  $\delta > 0$ . The bubble on Lucas trees is not essential in the sense that the first best is implementable when Lucas trees are priced at their fundamental value. Finally, in the white area the shortage of assets is so severe that the first best allocation is no longer implementable. Provided that  $\delta > 0$ , trees are priced at their fundamental value and fiat money is valued.

## 5 Inflation and asset prices

In this section we will look at the relationship between inflation and asset prices. The model is extended to allow the money supply to grow at the gross growth rate  $\gamma > \beta$ . The money supply per buyer in the DM of period  $t$  is  $\gamma^t$ . If  $\gamma > 1$  ( $\gamma < 1$ ), then money is injected (withdrawn) through lump-sum transfers (taxes) to buyers at the beginning of the CM.<sup>10</sup>

In accordance with the literature on monetary policy and asset prices reviewed in Section 1.1 we take the money growth rate as exogenously given; it is regarded as an additional constraint faced by the mechanism designer. This means that relative to existing studies we only change the mechanism in bilateral matches. In particular, we assume that the lump-sum taxes used to finance deflation cannot be used to finance returns on Lucas trees.<sup>11</sup>

We will focus on stationary equilibria where aggregate real balances are constant. Denote  $\vartheta_t$  the price of one unit of money in terms of the numéraire good in the CM of period  $t$ . Aggregate real balances per buyer is  $\phi = \vartheta_t \gamma^t$ . In a stationary equilibrium the rate of return for fiat money is  $\vartheta_{t+1}/\vartheta_t = \gamma^{-1}$ . Let  $m_t$  denote the money holdings of a buyer in period  $t$  as a fraction of the per capita money supply. The real value of the money holdings of a buyer at the beginning of the CM (before the transfers) is then  $\vartheta_t \gamma^t m_t = \phi m_t$ . This implies that the value functions are identical to (1) to (5) up to some constant

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<sup>10</sup>We assume that the government has enough coercive power to collect taxes in the CM, but it has no coercive power in the DM and it does not observe trading histories or asset holdings neither in the DM nor in the CM. There are alternative approaches to model deflation (Hu, Kennan, and Wallace, 2009; Andolfatto, 2010), where the buyers can choose not to participate the CM in order to avoid paying taxes.

<sup>11</sup>In Section 7 we discuss what happens if lump-sum taxes can be used to alter the return on Lucas trees. We also argue that one could relax the enforcement power of the government to levy lump sum taxes: this would amount to introducing a lower bound on the money growth rate without affecting the main insights of our analysis.

terms representing the lump-sum transfers (taxes) of money.

The maximization problem for buyers in the CM is

$$\max_{\hat{a} \geq 0, \hat{m} \geq 0} \left\{ -\gamma\phi\hat{m} - q\hat{a} + \beta V^b(\hat{a}, \hat{m}) \right\}, \quad (28)$$

where  $\hat{m}$  represents the buyer's money holding in period  $t+1$  as a fraction of the average money supply per buyer in the DM of  $t+1$ . Consequently,  $\gamma\phi\hat{m} = \vartheta_t\gamma^{t+1}\hat{m}$  represents the choice of real balances in the CM of  $t$ . Following the same reasoning as the one in the previous sections, the buyer's participation constraint in the CM becomes

$$-i\phi - [(r + \delta)q - 1]A + \sigma \{u(\bar{y}) - [(1 - \delta)q + 1]\bar{\tau}_a - \phi\bar{\tau}_m\} \geq 0, \quad (29)$$

where  $i = \frac{\gamma - \beta}{\beta}$  is the cost of holding real balances. (The quantity,  $i$ , would be the nominal interest rate paid by illiquid nominal bonds.) In the presence of a growing money supply, the cost of holding real balances includes not only the rate of time preference but also the rate at which the value of money declines over time,  $\gamma$ . Parallel to Lemma 2, a characterization of constrained-efficient outcomes involves two inequalities: an individual-rationality constraint for sellers, which is the same as (17), and an individual-rationality constraint for buyers which is given by (16) where  $r$  is replaced with  $i$ .

When money grows with a constant growth rate, whether  $y^*$  is implementable with putting liquidity premium on money alone depends on both the supply of assets,  $A$ , and the cost of holding money,  $i$ . If  $A \geq A_1$ , then, as shown earlier, no liquidity premium (either on money or trees) is necessary to implement  $y^*$ . If  $A < A_1$ , however, the critical threshold for  $y^*$  to be implementable by putting liquidity premium on money alone is

$$i^*(A) = \sigma[u(y^*) - v(y^*)] / [v(y^*) - (\delta + r)A / (1 + r)]. \quad (30)$$

It is straightforward to verify that  $i^*(A)$  is well-defined for  $A < A_1$ ,  $i^*(A)$  increases with  $A$ , and  $i^*(A) \rightarrow \infty$  as  $A$  approaches  $A_1$ . Moreover,  $i^*(A_3) = r$ .

**Proposition 4 (Inflation and asset prices)** *Let  $\gamma > \beta$  be the gross growth rate of money supply.*

*(i) Suppose that  $A \in [A_2, A_1]$  and that  $i > i^*(A)$ . An outcome,  $(y, q, \phi)$ , is constrained efficient only if  $y = y^*$ ,  $\ell \in [\ell^*(i, A), \bar{\ell}(A)]$ , where*

$$\ell^*(i, A) = \frac{i \left[ v(y^*) - \frac{1+r}{\delta+r}A \right] - \sigma[u(y^*) - v(y^*)]}{[i(1 - \delta) - (r + \delta)]A}. \quad (31)$$

(ii) Suppose that  $A < A_2$ . A first-best outcome,  $y = y^*$ , is implementable if and only if  $i \leq i^*(A)$ .

(ii.a) When  $i \leq i^*(A)$ , the outcome,  $(y, q, \phi)$ , is constrained efficient only if  $y = y^*$ ,  $\phi > 0$ , and  $\ell \leq \ell^*(i, A)$ .

(ii.b) When  $i \in (i^*(A), (r + \delta)/(1 - \delta))$ , the constrained-efficient outcome,  $(\bar{y}, q^*, \phi)$ , solves

$$\sigma[u(\bar{y}) - v(\bar{y})]/i = v(\bar{y}) - (1 + r)A/(r + \delta) = \phi.$$

(ii.c) When  $i = \frac{r+\delta}{1-\delta}$ , there are a continuum of constrained-efficient outcomes,  $(\bar{y}, \phi, \ell)$ , characterized by  $\frac{(1-\delta)\sigma[u(\bar{y})-v(\bar{y})]}{r+\delta} = v(\bar{y}) - \frac{1+r}{r+\delta}A = \phi + (1-\delta)A\ell$ .

(ii.d) When  $i > \frac{r+\delta}{1-\delta}$ , the constrained-efficient outcome,  $(\bar{y}, q, 0)$ , solves  $\frac{\sigma[u(\bar{y})-v(\bar{y})]}{r+\delta} = \frac{v(\bar{y}) - \frac{1+r}{r+\delta}A}{1-\delta} = A\ell$ .

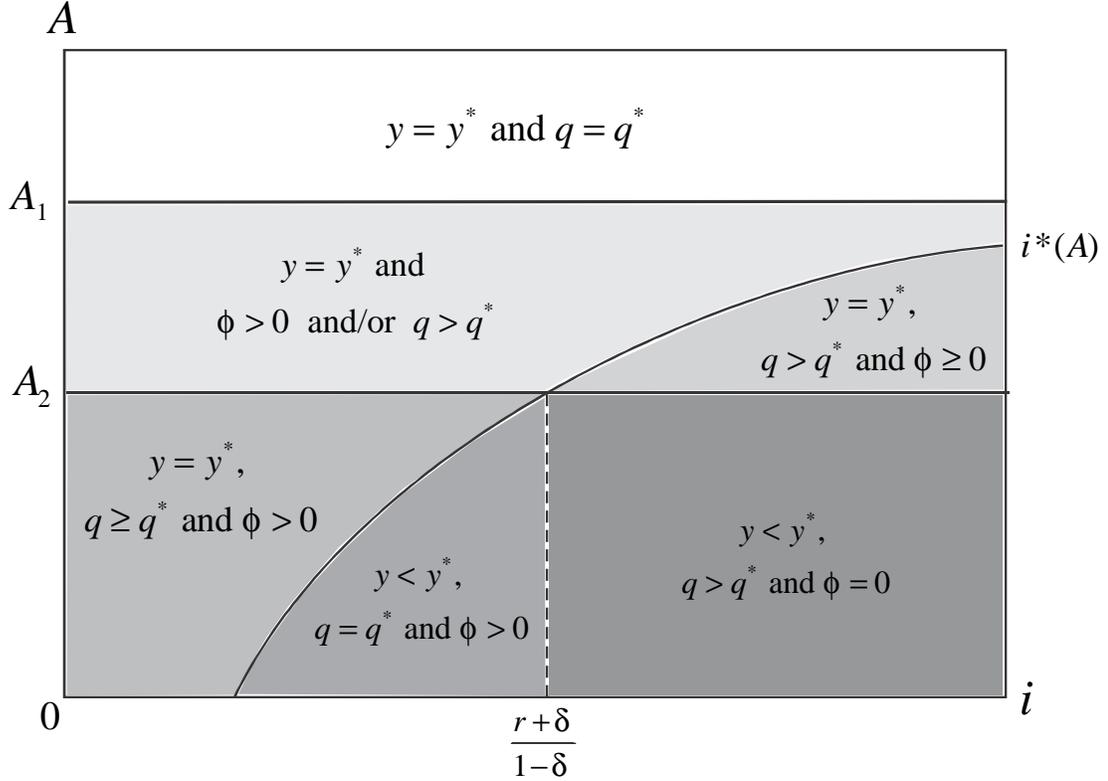


Figure 3: Bubbles and inflation

Figure 3 illustrates the main findings from Proposition 4. Departure of asset prices from their fundamental values occur when  $A < A_1$ , in the grey areas. When  $A \geq A_2$  and  $i \leq i^*(A)$  (the light grey area), the first-best level of output can be achieved with either valued fiat money (in which case  $q$  can be equal to  $q^*$ ) or a bubble on Lucas trees (in which case  $\phi$  can be zero) or both. When  $A \geq A_2$  and

$i > i^*(A)$ , a bubble on Lucas trees is essential as  $y^*$  cannot be implemented with valued fiat money alone. When  $i \leq i^*(A)$  and  $A < A_2$ , valued fiat money is essential as the first best cannot be implemented with a bubble on Lucas trees alone. Finally, when  $A < A_2$  and  $i > i^*(A)$ ,  $y^*$  is not implementable. If the effective cost of holding fiat money is less than the cost of holding a bubble on Lucas trees,  $i < \frac{r+\delta}{1-\delta}$ , then fiat money is valued and Lucas trees are priced at their fundamental value. Conversely, if  $i > \frac{r+\delta}{1-\delta}$ , then fiat money is not valued and the price of Lucas trees exhibits a bubble. These results show that a valued fiat money and a bubble on Lucas trees coexist only if  $y^*$  is implementable, or in the knife-edge case where  $i = \frac{r+\delta}{1-\delta}$ .<sup>12</sup> In contrast to the literature (such as Geromichalos, Licari, and Suarez-Lledo, 2007), Lucas trees are priced at their fundamental value when money is valued and  $y < y^*$  in any stationary, constrained-efficient equilibrium outcome, except for in the knife-edge case where  $i = \frac{r+\delta}{1-\delta}$ .

Proposition 4 also shows that the Friedman rule, defined as  $i = 0$ , is not necessary to implement  $y^*$ . From (30)  $i^*(0) = \frac{\sigma[u(y^*)-v(y^*)]}{v(y^*)} > 0$ . Therefore, for all  $i \leq \frac{\sigma[u(y^*)-v(y^*)]}{v(y^*)}$ ,  $y^*$  is implementable for any  $A$ .<sup>13</sup> If  $i$  is greater than  $i^*(0)$ , then  $y^*$  can still be implemented but only if there are enough Lucas trees to supplement the supply of fiat money.

In Figure 4 we represent the relationship between the total bubble on the trees,  $A\ell$ , and the cost of holding fiat money,  $i$ . The red areas on both panels correspond to the set of values for  $A\ell$  that are consistent with a constrained-efficient outcome. Consider first the right panel of Figure 4, when the supply of Lucas trees is abundant,  $A > A_2$ . In this case  $y = y^*$  can be implemented irrespective of the cost of holding fiat money. However, if  $i$  is above  $i^*(A)$ , then the set of values for  $A\ell$  shrinks as  $i$  increases, as the lower bound  $\ell^*(i, A)$ , expressed in (31), increases with  $i$ . Money becomes so costly to hold that  $y^*$  cannot be implemented with fiat money alone.

Consider next the left panel of Figure 4 when the supply of productive assets is scarce,  $A < A_2$ . If  $i$  is lower than  $i^*(A)$ , then  $y^*$  is implementable with a valued fiat money and with  $A\ell$  below the upper bound  $A\ell^*(i, A)$ , including  $A\ell = 0$  (meaning that the first best can be implemented with liquidity premium on fiat money alone). It is easy to verify that  $\ell^*(i, A)$  decreases with  $i$  (see the proof for a detailed derivation) when  $A < A_2$  and hence the maximum size for the bubble decreases with the cost of holding money. This result is consistent with the view that the monetary authority can “fuel”

<sup>12</sup>This is in sharp contrast with the existing literature that finds that bubbles do not emerge when the equilibrium achieves the first best, and whenever fiat money is valued there is a liquidity premium incorporated in the price of Lucas trees. See, e.g., Geromichalos, Licari, and Suarez-Lledo (2007).

<sup>13</sup>This result is similar to that in Hu-Kennan-Wallace (2009) and Rocheteau (2011). See Rocheteau (2011) for an interpretation of the threshold  $i^*(0)$ .

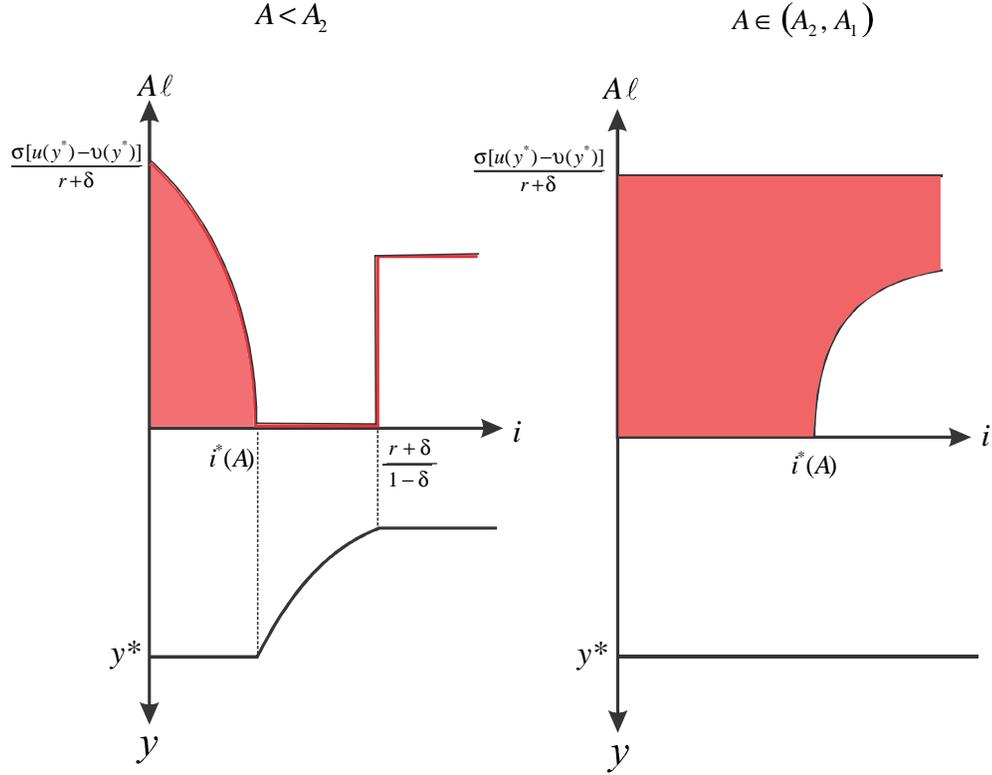


Figure 4: Inflation, output, and asset prices

asset price bubbles by lowering the cost of holding fiat money. When inflation is in some intermediate range, i.e.,  $i$  is between  $i^*(A)$  and  $\frac{r+\delta}{1-\delta}$ ,  $y^*$  is no longer implementable and Lucas trees are priced at their fundamental value. For such values for  $i$ , inflation has no effect on asset prices, but it reduces the value of fiat money and DM output. Moreover, Lucas trees have a higher rate of return than fiat money, which illustrates another instance of rate-of-return dominance.<sup>14</sup> If the cost of holding real balances is increased to the range  $i > \frac{r+\delta}{1-\delta}$ , then fiat money ceases to be valued and the a bubble is attached to the Lucas trees,  $q > q^*$ .

In summary, our model predicts a nonmonotonic relationship between inflation and asset prices. For low inflation rates, reducing  $i$  can generate asset price bubbles; For intermediate inflation rates, Lucas trees are priced at their fundamental value; For sufficiently high money growth rates, fiat money is not valued and Lucas trees are priced above their fundamental value.

<sup>14</sup>All these results contrast with the existing literature which finds that under arbitrary mechanisms asset prices increase with inflation and rates of return are equal across assets (e.g., Geromichalos, Licari, and Suarez-Lledo, 2007).

## 6 Resalability and asset prices

So far we have assumed that the transfer of assets in bilateral matches was seamless. In contrast, Kiyotaki and Moore (2005), Lagos (2010), and Lester, Postlewaite, and Wright (2011), among others, assume that not all assets are equally “resalable.” In the following we will capture a similar notion by assuming that there are a continuum of Lucas trees indexed by  $j \in [0, 1]$ , and ranked according to the degree of difficulty for a seller to authenticate the asset. The supply of a measurable subset of assets,  $J \subset [0, 1]$ , is  $\int_J A(j) dj$  and all assets are in equal supply,  $A(j) = A$ . Each type of asset can be counterfeited at no cost, but sellers can invest in a costly technology to recognize assets. The cost to authenticate an asset in the interval  $[0, x]$  is  $C(x)$ . It is implicit in this formulation that if a seller can recognize asset  $j$ , then he can also recognize any asset  $j' < j$ . Moreover, we assume that the composition (but, of course, not the quality of the asset of each type) of the buyer’s portfolio is common knowledge within the match.

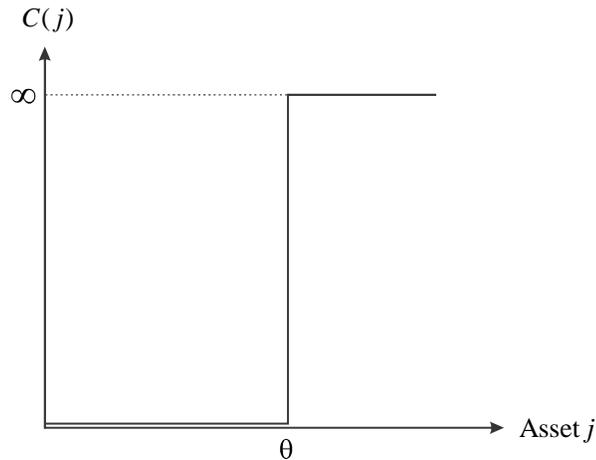


Figure 5: Cost function to authenticate assets

To simplify the exposition we assume a cost function of the form  $C(j) = 0$  for all  $j \leq \theta$  and  $C(j) = +\infty$  for all  $j \in (\theta, 1]$  and for some exogenous  $\theta < 1$ . See Figure 5. (In Appendix B we extend the results to a more general cost function). This assumption means that a fraction  $\theta$  of assets can be authenticated at no cost while the remaining assets cannot be authenticated. The former assets are called *recognizable* while the latter are *unrecognizable*. Unrecognizable assets are not accepted in trade, and buyers can overstate their holdings of such assets. It follows immediately that unrecognizable assets must be priced at their fundamental value. The price of a recognizable asset is  $q(j) = q^* + \ell(j)$  and the

transfer of assets in a match is  $\int_0^\theta \tau_a(j) dj$ . Following the same reasoning as above, the necessary and sufficient conditions for a constrained-efficient outcome are

$$\begin{aligned}
 -i\phi - (r + \delta) \int_0^\theta \ell(j) A dj + \sigma [u(y) - v(y)] &\geq 0 \\
 \phi + (1 - \delta) \int_0^\theta \ell(j) A dj + \frac{1+r}{\delta+r} \theta A - v(y) &\geq 0.
 \end{aligned}$$

We focus on equilibria where  $\ell(j)$  is constant across recognizable assets. The model is then identical to the one of the previous section where the supply of liquid assets is now  $\theta A$ . The results of Proposition 4 apply. If  $\theta A < A_2$ , then  $y^*$  is implementable and an asset price bubble can emerge only if  $i < i^*(\theta A)$ . If  $i > i^*(\theta A)$  and  $i < (r + \delta)/(1 - \delta)$ , then  $y < y^*$  and all assets are priced at their fundamental values. So for a given interest rate,  $i$ , a lower set of recognizable assets makes it more likely that the first best is not implementable and asset prices do not exhibit a bubble.

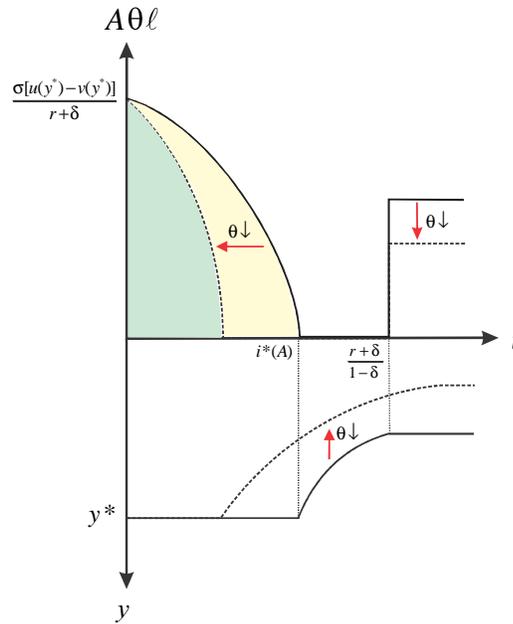


Figure 6: Effects of a decrease in asset recognizability

Some economists have interpreted the 2007-08 financial crisis as a shock that reduced the effective liquidity of the economy. Assets such as mortgage-backed securities that were used as collateral in OTC and repo markets became suddenly illiquid.<sup>15</sup> We capture such a narrative in our model by a shock that

<sup>15</sup>In an interview to the Wall Street journal (09/24/2011), Robert Lucas argued that "the shock came because complex mortgage-related securities minted by Wall Street and certified as safe by rating agencies had become part of the effective liquidity supply of the system. All of a sudden, a whole bunch of this stuff turns out to be crap".

reduces the set of recognizable assets, i.e.,  $\theta$  falls. The effects of the shock are illustrated in Figure 6. If a smaller set of assets are recognizable, then the maximum interest rate below which the first best is implementable,  $i^*(A)$ , falls. Moreover, the maximum size for the aggregate bubble,  $\theta A\ell$ , when  $i < i^*(A)$  shrinks. Graphically, the downward sloping curve in the top panel of Figure 6 shifts to the left. This finding suggest that a shock that reduces the supply of liquid assets can generate an optimal burst of the asset price bubble. Moreover, if monetary policy is chosen optimally, such an event can also lead to lower interest rates. In contrast, if the first best is not implementable, then the burst of the bubble is associated with a fall of output. Graphically, the upward-sloping curve in the bottom panel of Figure 6 shifts upward. For sufficiently high interest rates such that fiat money is not valued, a reduction in the set of liquid assets also leads to a lower liquidity premium on Lucas trees and a reduction in output.

In the Appendix B we generalize the technology to authenticate assets as follows. For all  $x \in [0, \theta_0]$  for some  $\theta_0 > 0$ ,  $C(x) = 0$ . So there is still a set of assets that can be authenticated at no cost. In contrast to the previous formulation, however,  $C'(x) > 0$ , and  $C''(x) > 0$  for all  $x > \theta_0$ . Therefore, sellers can invest in a costly technology in order to be able to authenticate assets beyond the measure,  $\theta_0$ , that are perfectly recognizable. We establish that if the supply of the assets that can be authenticated at no cost,  $\theta_0 A$ , is greater than the threshold  $A_2$ , then the first best is implementable without fiat money. In contrast, if  $\theta_0 A < A_2$  then fiat money is essential. If the cost of holding fiat money is lower than a threshold,  $i^*(\theta_0 A)$ , then  $y^*$  is implementable and agents do not invest in a costly technology to recognize assets, i.e.,  $\theta < \theta_0$ . If the inflation rate is too large for the first best to be implementable, then it is optimal for sellers to invest in a costly information technology in order to expand the set of recognizable assets,  $\theta > \theta_0$ . This finding is in accordance with an old idea according to which money saves information costs (e.g., Brunner and Meltzer, 1971). Inflation, by reducing the real value of fiat money—a perfectly recognizable asset—induces sellers to spend resources to assess the quality of alternative assets. Even though the set of liquid assets expands, Lucas trees do not pay a liquidity premium. The reasoning is the same as before. Given that the cost of holding fiat money is small relative to the depreciation rate of Lucas trees,  $i < (r + \delta)/(1 - \delta)$ , it is cheaper to attach a liquidity premium to fiat money. Finally, if the money growth rate is larger than a threshold, then fiat money is no longer valued and recognizable assets exhibit a bubble.

## 7 Concluding remarks

We studied a model of an OTC market where agents trade assets to finance consumption opportunities in bilateral matches. In contrast to the existing literature the trading mechanism in the OTC market is chosen to be socially optimal taking as given the frictions in the environment and monetary policy. We derived a set of new results regarding the relationship between asset prices and monetary policy. First, we obtained a rate-of-return dominance result, which states that money is essential even though the (short-lived) Lucas trees pay a positive return. Second, we obtained new predictions for the relationship between monetary policy and asset prices. Loose monetary policy by making liquidity abundant can fuel bubbles. In contrast, when liquidity is scarce, assets are priced at their fundamental value in any constrained-efficient monetary equilibrium. Finally, we also showed that the burst of an asset price bubble can be an optimal response to a shock that reduces the set of assets that can be readily authenticated in the OTC market. These results are obtained with a few assumptions and in the following we discuss their roles.

First, to keep the supply of Lucas trees stationary, we assumed that at each period new Lucas trees flow into the economy in a lump-sum fashion to replace the depreciated ones, and the replacement trees are identical to the original ones. If agents could distinguish trees according to their ages, then any allocation that could be implemented with a valued fiat money under constant money supply could also be achieved by assigning a liquidity premium to date-0 trees. To see this, notice that any implementable equilibrium outcome,  $(y, q, \phi)$ , with  $q = q^*$  and  $\phi > 0$  can also be implemented with the following scheme: set the value of fiat money to  $\phi' = 0$  and let the price of date-0 trees be  $q'_t = q^* + \ell_t$  with  $A\ell_t = \phi/(1-\delta)^t$ . As the supply of date-0 trees shrinks over time, the price of those trees increases so as to keep the size of the aggregate bubble,  $A(1-\delta)^t\ell_t$ , constant and equal to  $\phi$ .

We assumed that lump-sum taxes were available to engineer a deflation. If the government did not have enough power to enforce the payment of such taxes, then there would be a lower bound on the nominal interest rate, i.e.,  $i \geq r$ . In this case Proposition 4 is still valid. In particular, as long as  $A \leq A_2$  is not too small,  $y^*$  is implementable with  $i = r$  and hence there is a range of inflation rates for which a bubble on asset prices is consistent with the optimal mechanism. If lump-sum taxes could also be used to subsidize the return on Lucas trees, then one could make Lucas trees sufficiently valuable to engineer a first-best outcome.

Finally, our results can be easily extended to a setting with multiple assets that differ in their

dividends and depreciation rates. In this context, fiat money can be regarded as a Lucas tree with zero dividend and with a depreciation rate equal to  $\delta = 1 - 1/\gamma$ . If aggregate liquidity is scarce, then it is always useful to add an asset with a lower depreciation rate than the ones of existing assets, regardless of dividends.

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## Appendix A. Proofs

**Proof of Lemma 1** Given a trading mechanism in the DM,  $o$ , the problem of a buyer holding  $a$  units of Lucas trees and  $m$  units of fiat money when entering the CM is:

$$W^b(a, m) = \max_{\hat{a} \geq 0, \hat{m} \geq 0} \left\{ \delta q A + \phi m + [(1 - \delta)q + 1]a - \phi \hat{m} - q \hat{a} + \beta V^b(\hat{a}, \hat{m}) \right\}, \quad (32)$$

where  $\hat{a}$  and  $\hat{m}$  denote the desirable asset holdings for the next DM and  $V^b(a, m)$  is the expected lifetime utility of a buyer holding  $a$  units of trees and  $m$  units of fiat money upon entering the DM conditional on the trading mechanism  $o$ . This uses the budget constraint, which dictates that the net consumption of the CM good is  $z = \delta q A + \phi m + [(1 - \delta)q + 1]a - \phi \hat{m} - q \hat{a}$ , where the first term is the lump-sum endowment of Lucas trees to replace the depreciated, the second and third term represent the value of the buyer's portfolio upon entering the CM, and the last two terms correspond to the cost of the end-of-period portfolio. The value of one unit of Lucas tree carried into the CM includes both the resale price times its survival probability and its dividend,  $(1 - \delta)q + 1$ . From (32) the optimal choice of  $\hat{a}$  and  $\hat{m}$  is independent of the buyer's portfolio,  $(a, m)$ , when entering the CM. Hence,

$$W^b(a, m) = \phi m + [(1 - \delta)q + 1]a + W^b(0, 0).$$

Similarly, the value function of a seller in the CM is  $W^s(a, m) = \phi m + [(1 - \delta)q + 1]a + \beta V^s$ , where  $V^s$  is the expected life-time utility for the seller upon entering the DM. Recall that sellers do not to carry assets across periods.

Now we turn to value functions for the DM. Assuming that agents report their portfolio truthfully and that they accept the trades proposed by the mechanism, the Bellman's equation for a buyer holding the portfolio  $(a, m)$  when entering the DM is

$$V^b(a, m) = \sigma \left\{ u[y(a, m)] + W^b[a - \tau_a(a, m), m - \tau_m(a, m)] \right\} + (1 - \sigma) W^b(a, m). \quad (33)$$

The buyer meets a seller with probability  $\sigma$ , in which case he consumes  $y$  units of goods and delivers  $\tau_a$  units of tree and  $\tau_m$  units of money; this transaction is reflected in the first term of the RHS in (33). The terms of trade,  $(y, \tau_a, \tau_m)$ , are generated by the trading mechanism,  $o$ , and they depend on the (truthfully) announced portfolio of the buyer. With complement probability,  $1 - \sigma$ , the buyer is unmatched and no trade takes place in the DM. Similarly, the Bellman's equation for a seller at the beginning of the period is

$$V^s = \sigma \left\{ -v[y^s(A, 1)] + W^s[\tau_a^s(A, 1), \tau_m^s(A, 1)] \right\} + \beta V^s. \quad (34)$$

Notice that in simple equilibria all buyers hold the same portfolio  $(A, 1)$ . The lemma follows directly from (32)-(34).

**Proof of Proposition 1.** (*Necessity*) In order to check the sequential rationality of buyers' portfolio choices in the CM, substitute  $V^b(a, m)$  by its expression given by (1) into (32) to obtain (omitting constant terms):

$$(A, 1) \in \arg \max_{a \geq 0, m \geq 0} \{-r\phi m - [(r + \delta)q - 1]a + \sigma \{u[y(a, m)] - \phi\tau_m(a, m) - [(1 - \delta)q + 1]\tau_a(a, m)\}\}. \quad (35)$$

Consider the following deviations:

(i) The seller, instead of taking the proposed trade in the DM, could go to the CM without production in the DM, and then follow the equilibrium behavior. This deviation is not profitable if and only if

$$V^s \geq \beta V^s \text{ if and only if } V^s \geq 0,$$

which from (4) is equivalent to (11).

(ii) The buyer, instead of buying back the assets in the CM, could sell all his assets in this period, consuming nothing in the DM and resume equilibrium behavior afterwards. From (35) this deviation which consists in setting  $(a, m) = (0, 0)$  is not profitable if and only if (10) holds.

(iii) If  $q < \frac{1}{r+\delta}$ , then from (35) the buyer's choice of asset holdings is unbounded.

(*Sufficiency*) Clearly, the solutions to the maximization problems (13) and (14) both exist and each solution has a unique  $y(a, m)$ . Although  $\tau_a$  and  $\tau_m$  may not be uniquely determined, we will select the solution such that  $\tau_m(a, m) = m$  if it exists and  $\tau_a(a, m) = 0$  otherwise. This gives us a well-defined mechanism  $o = (y, \tau_a, \tau_m)$ .

First,  $o(A, 1) = (\bar{y}, \bar{\tau}_a, \bar{\tau}_m)$ . To see this notice that (13) is the dual problem of (6)-(8), which defines the core of a pairwise meeting. Because  $(\bar{y}, \bar{\tau}_a, 1) \in \mathcal{C}(A, 1, q, \phi)$ ,  $(\bar{y}, \bar{\tau}_a, \bar{\tau}_m)$  is also a solution to (13).

Next, we propose equilibrium strategies and beliefs. Buyer strategies are as follows: in the DM the buyer always tells the truth (for any portfolio) in the first stage and responses with yes; the buyer always leaves the CM with  $(A, 1)$ . Seller strategies are as follows: in the DM always response with yes. Given the mechanism,  $o$ , and the strategies the value functions are given by (32)-(34).

Now we show that the strategies are optimal. From (13) and (14),  $u(y) - \phi\tau_m - [(1 - \delta)q + 1]\tau_a \geq 0$  and  $-v(y) + \phi\tau_m + [(1 - \delta)q + 1]\tau_a \geq 0$ . Indeed if  $a \geq A$  and  $m \geq 1$  then the seller enjoys a payoff at

least as large as the one he would get if  $a = A$  and  $m = 1$ , and from (11) this payoff is nonnegative; if  $a < A$  or  $m < 1$ , then the seller's payoff is at least as large as the one he would get if  $y = \tau_a = \tau_m = 0$ . Therefore, it is optimal for both the buyer and the seller to say yes to the proposed trade. Moreover, from (35) and the definition of  $o$  in (13)-(14), the buyer's problem in the CM is

$$\max_{a \geq 0, m \geq 0} -r\phi m - [(r + \delta)q - 1]a + \sigma \{u(\bar{y}) - \phi\bar{\tau}_m - [(1 - \delta)q + 1]\bar{\tau}_a\} \mathbb{I}_{\{m \geq 1, a \geq A\}}.$$

From (10) it is optimal for buyers to leave the CM with  $(A, 1)$ .

It remains to show that truth-telling is always optimal. Denote  $(a', m')$  the buyer's announced portfolio and  $(a, m)$  his true portfolio. Given that by assumption buyers cannot overstate their asset holdings, the buyer's optimal announcement is

$$(a', m') \in \arg \max_{a' \leq a, m' \leq m} \{u[y(a', m')] - \phi\tau_m(a', m') - [(1 - \delta)q + 1]\tau_a(a', m;)\}.$$

From the definition of  $o$  in (13)-(14),

$$(a', m') \in \arg \max_{a' \leq a, m' \leq m} [u(\bar{y}) - \phi\bar{\tau}_m - [(1 - \delta)q + 1]\bar{\tau}_a] \mathbb{I}_{\{m' \geq 1, a' \geq A\}}.$$

From (10)  $u(y) - \phi\bar{\tau}_m - [(1 - \delta)q + 1]\bar{\tau}_a \geq 0$ . Therefore the buyer's surplus is (weakly) increasing in his announced asset holdings and it is optimal to announce  $(a', m') = (a, m)$ .  $\square$

**Proof of Lemma 2.** ( $\Leftarrow$ ) Suppose that  $(y^*, q, \phi)$  satisfies (16) and (17). Consider two cases. First suppose that  $v(y^*) \geq \phi$ . Let  $\bar{\tau}_a = \frac{v(y^*) - \phi}{[(1 - \delta)q + 1]}$ . Then  $(y^*, \bar{\tau}_a, 1, q, \phi)$  is implementable. Now suppose that  $v(y^*) < \phi$ . Let  $\bar{\tau}_m = \frac{v(y^*)}{\phi}$ . Then  $(y^*, 0, \bar{\tau}_m, q, \phi)$  is implementable.

Suppose that  $(\bar{y}, q, \phi)$  solves the maximization problem and  $\bar{y} < y^*$ . If  $\phi + [(1 - \delta)q + 1]A - v(\bar{y}) > 0$ , then there exists some  $y' \in (\bar{y}, y^*)$  such that  $(y', q, \phi)$  satisfies (16) and (17), a contradiction to the optimality of  $\bar{y}$ . Hence, (17) holds at equality and  $(\bar{y}, A, 1, q, \phi)$  is implementable (notice that  $\bar{\tau}_a = A$  is necessary for implementability because of the pairwise core requirement). Now any implementable outcome  $(y'', \tau''_a, \tau''_m, q'', \phi'')$  will satisfy the constraints in the maximization problem and hence  $y'' \leq \bar{y}$  (notice that (10) and (11) imply (16) and (17)).

( $\Rightarrow$ ) Suppose that  $(\bar{y}, \bar{\tau}_a, \bar{\tau}_m, q, \phi)$  is a constrained-efficient outcome. Then the constraints in the maximization problem hold. Moreover, if there is a triple  $(y', q', \phi')$  that maximizes the problem in the lemma with higher DM output than  $\bar{y}$ , then by the above arguments we know that there are payment schemes so that  $(y', q', \phi')$  is also implementable. This is a contradiction to the constrained efficiency of the equilibrium outcome  $(\bar{y}, \bar{\tau}_a, \bar{\tau}_m, q, \phi)$ .

**Proof of Proposition 2** (i) Consider an equilibrium with  $\ell = 0$ , i.e.,  $q = q^* = (r + \delta)^{-1}$ , and  $y = y^*$ . The condition (16) is always satisfied. The condition (17) is equivalent to  $A \geq A_1$ .

(ii) Suppose that  $A < A_1$ , i.e., the first-best allocation cannot be implemented with  $q = q^*$ . From (16)-(17),  $y = y^*$  can be implemented with  $\phi = 0$  if and only if there is a  $q \geq q^*$  such that

$$\frac{v(y^*) - A}{1 - \delta} \leq qA \leq \frac{\sigma[u(y^*) - v(y^*)] + A}{r + \delta}.$$

This requires  $A \geq A_2$ . Moreover, the second inequality gives the expression for  $\bar{\ell}(A)$ .

(iii) Since  $y = y^*$  is not implementable,  $y < y^*$  and (16) and (17) must hold at equality (otherwise  $q$  and  $y$  could be raised). Therefore,  $q$  is the solution to  $[(r + \delta)q - 1]A = \sigma[u(y) - v(y)]$ , where  $y = v^{-1}[(1 - \delta)q + 1]A$ .  $\square$

**Proof of Proposition 3.** Consider any output level  $y$ . From Lemma 2  $y$  is feasible in the maximization problem if there exists  $(\ell, \phi)$  such that the following constraints hold:

$$\phi + (1 - \delta)A\ell \geq v(y) - \frac{1 + r}{\delta + r}A \quad (36)$$

$$\phi + \left(1 + \frac{\delta}{r}\right)A\ell \leq \frac{\sigma[u(y) - v(y)]}{r}. \quad (37)$$

For each output level  $y$ , let  $IRS$  denote the line in the  $(\ell, \phi)$ -space corresponding to (36) with equality and let  $IRB$  denote the line corresponding to (37) with equality. These two lines are represented in Figure 7. It is easy to check that for all  $\delta > 0$  the absolute value of the slope of  $IRB$  is greater than the absolute value of the slope of  $IRS$ . Define the following intercepts of  $IRS$  and  $IRB$ :

$$\begin{aligned} \phi_s(y) &= v(y) - (1 + r)A/(\delta + r), & \ell_s(y) &= \left[v(y) - \frac{1 + r}{\delta + r}A\right] / [(1 - \delta)A], \\ \phi_b(y) &= \sigma[u(y) - v(y)]/r, & \ell_b(y) &= \sigma[u(y) - v(y)] / [(r + \delta)A]. \end{aligned}$$

$\phi_s(y)$  (resp.  $\phi_b(y)$ ) and  $\ell_s(y)$  (resp.  $\ell_b(y)$ ) are the intercepts of the  $IRS$  (resp.  $IRB$ ) with the  $\phi$ -axis and the  $\ell$ -axis, respectively. The output level,  $y$ , is feasible for some pair  $(\ell, \phi)$  if and only if  $\phi_s(y) \leq \phi_b(y)$  or  $\ell_s(y) \leq \ell_b(y)$ . Graphically, in Figure 7,  $(\ell, \phi)$  is located in the light green area. It turns out that verifying  $\phi_s(y) \leq \phi_b(y)$  is sufficient.

**Lemma 3** (a) Suppose  $\delta > 0$ . If  $\ell_s(y) \leq \ell_b(y)$ , then  $\phi_s(y) \leq \phi_b(y)$ .

(b) Suppose that  $\delta = 0$ .  $\phi_s(y) \leq \phi_b(y)$  if and only if  $\ell_s(y) \leq \ell_b(y)$ .

As illustrated in the left and middle panels of Figure 7, any implementable  $y$  can be implemented with  $\phi > 0$  and  $\ell = 0$ . Therefore, the constrained-efficient outcome has  $y = y^*$  if and only if  $\phi_s(y^*) \leq \phi_b(y^*)$ .

Moreover,

$$\phi_s(y^*) \leq \phi_b(y^*) \Leftrightarrow v(y^*) - \frac{1+r}{\delta+r}A \leq \frac{\sigma}{r} [u(y^*) - v(y^*)] \Leftrightarrow A \geq A_3.$$

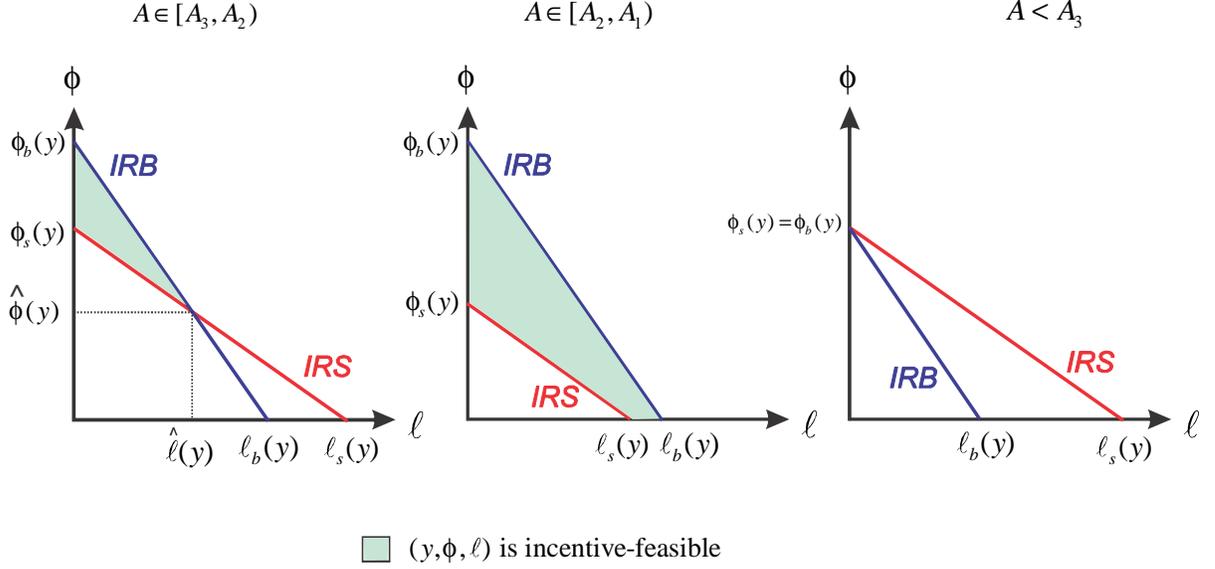


Figure 7: Illustration of proof of Proposition 3

(i)  $A \in [A_2, A_1)$ . Notice that  $A \geq A_2$  if and only if  $\ell_s(y^*) \leq \ell_b(y^*)$ . By Lemma 3 we know that  $\phi_s(y^*) \leq \phi_b(y^*)$ . Hence, *IRS* and *IRB* have no intersection within the region  $\mathbb{R}_+ \times \mathbb{R}_+$ . See middle panel of Figure 7. Any combination of  $(\ell, \phi) \in \mathbb{R}_+ \times \mathbb{R}_+$  that lies above *IRS*( $y^*$ ) and below *IRB*( $y^*$ ) will make  $y^*$  feasible. Thus, any liquidity premium on trees  $\ell$  below  $\ell_b(y^*) = \bar{\ell}(A)$  is consistent with  $y^*$ .

(ii)  $A \in [A_3, A_2)$ . In this case,  $\ell_s(y^*) > \ell_b(y^*)$  but  $\phi_s(y^*) \leq \phi_b(y^*)$ .  $y^*$  is still implementable, but *IRS* and *IRB* intersect at a point  $(\hat{\ell}, \hat{\phi}) \in \mathbb{R}_+ \times \mathbb{R}_+$ , where

$$\hat{\ell}(y^*) = \frac{\sigma[u(y^*) - v(y^*)] - r \left[ v(y^*) - \frac{1+r}{\delta+r}A \right]}{(1+r)\delta A} \geq 0,$$

$$\hat{\phi}(y^*) = \frac{(r+\delta)v(y^*) - (1-\delta)\sigma[u(y^*) - v(y^*)] - (1+r)A}{r(1+\delta)} > 0.$$

See left panel of Figure 7. Notice that  $\hat{\ell}(y^*) \geq 0$  because  $A \geq A_3$  and  $\hat{\phi} > 0$  because  $A < A_2$ . For  $\ell \leq \hat{\ell}(y^*)$ , there always exists a range of  $\phi$  so that (36) and (37) are both satisfied with  $y = y^*$ .

(iii)  $A < A_3$  and  $\delta > 0$ . Then  $y^*$  is not feasible, and the constrained-efficient outcome has the output level equal to the maximum level so that  $\phi_s(y) \leq \phi_b(y)$ . Because of strict concavity of  $u$  and convexity of  $v$ , the maximum  $y = y^{\max}(A)$  solves

$$\frac{\sigma}{r} [u(y) - v(y)] = v(y) - \frac{1+r}{\delta+r} A.$$

Moreover, there is a unique pair,  $(\ell, \phi)$ , such that  $y = y^{\max}(A)$  is feasible:  $\ell = 0$  and  $\phi = \phi_s(y^{\max}) = \phi_b(y^{\max})$ . See right panel of Figure 7.

(iv)  $A < A_3$  and  $\delta = 0$ . The determination of optimal  $y$  is the same as (iii); however, here at the optimal  $y$ , the two lines  $IRS$  and  $IRB$  coincide, as pointed out by Lemma 3. Thus, any combination of  $(\ell, \phi)$  on that line will make the optimal  $y$  feasible.  $\square$

**Proof of Proposition 4.** The proof is analogous to the proof of Proposition 3. A level of DM output,  $y$ , is feasible if there exists a pair,  $(\ell, \phi)$ , such that:

$$\phi + (1 - \delta)A\ell \geq v(y) - \left(\frac{1+r}{\delta+r}\right) A \quad (38)$$

$$\phi + \left(\frac{r+\delta}{i}\right) A\ell \leq \frac{\sigma [u(y) - v(y)]}{i}. \quad (39)$$

For each output level  $y$ , let  $IRS$  denote the line in the  $(\ell, \phi)$ -space corresponding to (38) at equality, and let  $IRB$  denote the line corresponding to (39) at equality. Define the following intercepts of  $IRS$  and  $IRB$ :

$$\begin{aligned} \phi_s(y) &= v(y) - \left(\frac{1+r}{\delta+r}\right) A, \quad \ell_s(y) = \left[ v(y) - \left(\frac{1+r}{\delta+r}\right) A \right] / [(1-\delta)A], \\ \phi_b(y) &= \sigma [u(y) - v(y)] / i, \quad \ell_b(y) = \sigma [u(y) - v(y)] / [(r+\delta)A]. \end{aligned}$$

$\phi_s(y)$  (resp.,  $\phi_b(y)$ ) and  $\ell_s(y)$  (resp.,  $\ell_b(y)$ ) are the intercepts of the line  $IRS$  (resp.,  $IRB$ ) with the  $\phi$ -axis and the  $\ell$ -axis, respectively. The output level,  $y$ , is feasible if and only if  $\phi_s(y) \leq \phi_b(y)$  or  $\ell_s(y) \leq \ell_b(y)$ . See Figure 8.

**Lemma 4** (a) Suppose  $i < \frac{r+\delta}{1-\delta}$ . If  $\ell_s(y) \leq \ell_b(y)$ , then  $\phi_s(y) < \phi_b(y)$ .

(b) Suppose that  $i = \frac{r+\delta}{1-\delta}$ .  $\phi_s(y) \leq \phi_b(y)$  if and only if  $\ell_s(y) \leq \ell_b(y)$ .

(c) Suppose that  $i > \frac{r+\delta}{1-\delta}$ . If  $\phi_s(y) \leq \phi_b(y)$ , then  $\ell_s(y) < \ell_b(y)$ .

For  $i \leq \frac{r+\delta}{1-\delta}$ , the constrained-efficient outcome has  $y = y^*$  if and only if  $\phi_s(y^*) \leq \phi_b(y^*)$ , that is,  $A \geq A_1$  or  $i \leq i^*(A)$ . For all  $i > \frac{r+\delta}{1-\delta}$ , the constrained-efficient outcome has  $y = y^*$  if and only if

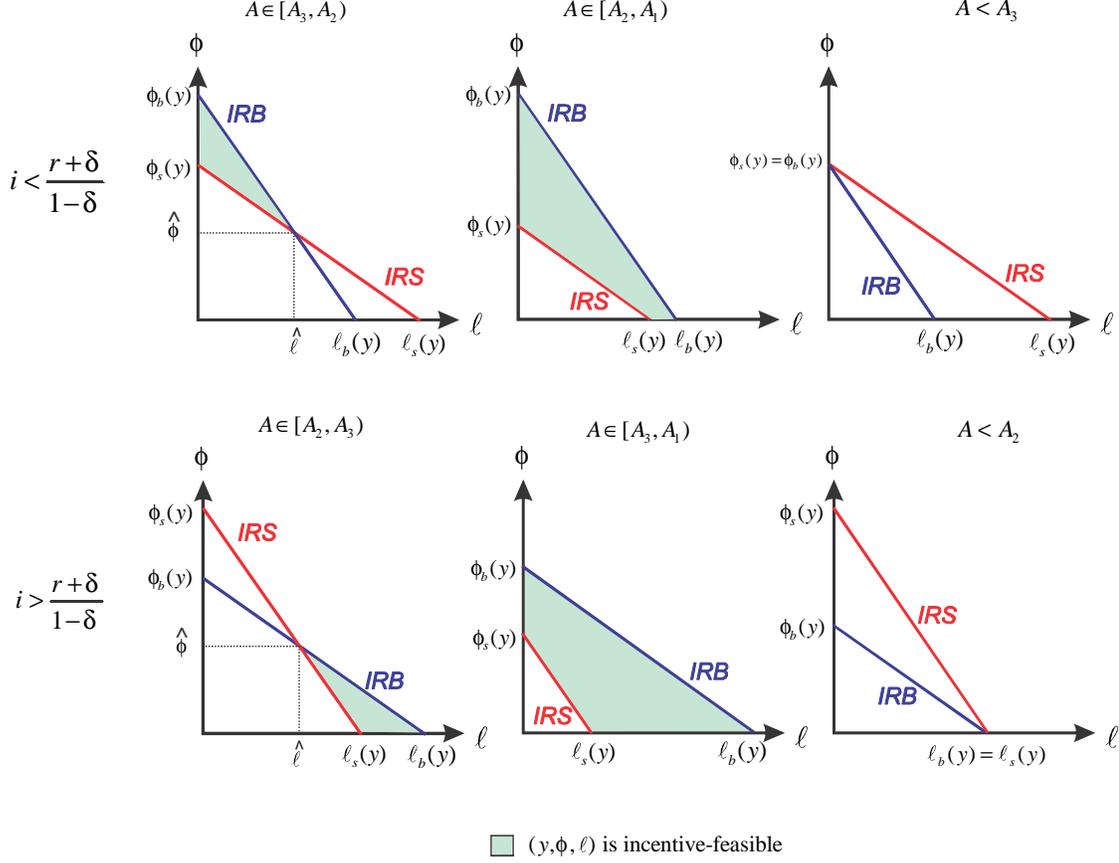


Figure 8: Illustration of proof of Proposition 4

$\ell_s(y^*) \leq \ell_b(y^*)$ , that is,  $A \geq A_2$ . It turns out that  $i^*(A) < \frac{r+\delta}{1-\delta}$  if and only if  $A < A_2$ . Thus, for  $A < A_2$ ,  $y^*$  is implementable if and only if  $i \leq i^*(A)$ .

(i)  $A \in [A_2, A_1)$ . Then  $y^*$  is feasible. Recall from the proof of Proposition 3 that  $A \geq A_2$  if and only if  $\ell_s(y^*) \leq \ell_b(y^*)$ . When  $i \leq i^*(A)$ , the upper bound for  $\ell$  is  $\ell_b(y^*) = \bar{\ell}(A)$  and the lower bound is zero, as in Proposition 3 (i); see middle panels on both top and bottom rows in Figure 8. When  $i > i^*(A)$ , we have the left panel of bottom row in Figure 8. In this case,  $\phi_s(y^*) > \phi_b(y^*)$  and  $IRS$  and  $IRB$  have an intersection within the region  $\mathbb{R}_+ \times \mathbb{R}_+$ . The  $\ell$ -coordinate of the intersection is

$$\ell^*(i, A) = \frac{i \left[ v(y^*) - \frac{1+r}{\delta+r} A \right] - \sigma [u(y^*) - v(y^*)]}{[i(1-\delta) - (r+\delta)]A}. \quad (40)$$

$\ell^*(i, A) > 0$  because  $A \geq A_2$  and  $i > i^*(A) \geq \frac{r+\delta}{1-\delta}$ . The upper bound for  $\ell$  is still  $\bar{\ell}(A)$ . Thus, only for  $\ell \in [\ell^*(i, A), \bar{\ell}(A)]$  there exists  $\phi$  such that  $y^*$  is feasible. Notice that

$$\frac{\partial}{\partial i} \ell^*(i, A) = \frac{-(r+\delta) \left[ v(y^*) - \frac{1+r}{\delta+r} A \right] + (1-\delta)\sigma[u(y^*) - v(y^*)]}{A[i(1-\delta) - (r+\delta)]^2}$$

and hence  $\frac{\partial}{\partial i} \ell^*(i, A) \geq 0$  if and only if  $A \geq A_2$ . So the lower bound for  $\ell$  is increasing in  $i$ , and

$$\lim_{i \rightarrow \infty} \ell^*(i, A) = \frac{1}{(1-\delta)A} \left[ v(y^*) - \frac{1+r}{\delta+r} A \right] = \ell_s(y^*).$$

(ii)  $A < A_2$ . In this case,  $\ell_s(y^*) > \ell_b(y^*)$ .  $y^*$  is implementable if and only if  $i \leq i^*(A) < \frac{r+\delta}{1-\delta}$ .

(ii.a)  $i \leq i^*(A)$ . See left panel of top row in Figure 8. Then  $y^*$  is implementable. Because  $\phi_s(y^*) \leq \phi_b(y^*)$  and  $\ell_s(y^*) > \ell_b(y^*)$ , the two lines  $IRS$  and  $IRB$  have an intersection within the region  $\mathbb{R}_+ \times \mathbb{R}_+$  and the  $(\ell, \phi)$  pairs that make  $y^*$  feasible are those on the left of that intersection. The  $\ell$ -coordinate of the intersection is still given by  $\ell^*(i, A)$  as in (40). Thus, the region where there are  $\phi$ s that make  $y^*$  feasible for  $\ell$  is  $\ell \in [0, \ell^*(i, A)]$ . Notice that because  $A < A_2$ ,  $\frac{\partial}{\partial i} \ell^*(i, A) < 0$  here.

(ii.b)  $i \in (i^*(A), \frac{r+\delta}{1-\delta})$ . See right panel on top row in Figure 8. Here  $y^*$  is not implementable. Moreover, it is easy to verify that the region within  $\mathbb{R}_+ \times \mathbb{R}_+$  for which both (38) and (39) are satisfied shrinks as  $y$  increases for all  $y < y^*$ . Therefore, by Lemma 4, the maximum  $y$  for which that region is nonempty satisfies  $\phi_s(y) = \phi_b(y)$ . This equality determines the output level  $\bar{y}$ . Moreover, by Lemma 4, this implies that the constrained efficient outcome has  $\ell = 0$  and the price for money  $\phi$  is determined by  $\phi = \phi_s(\bar{y})$ .

(ii.c)  $i = \frac{r+\delta}{1-\delta}$ . Here  $y^*$  is not implementable. As in (ii.b), by Lemma 4, the optimal output level is determined by  $\phi_s(y) = \phi_b(y)$  or  $\ell_s(y) = \ell_b(y)$  (in this case the two conditions are equivalent). This condition gives a unique solution to  $\bar{y}$ . Moreover, for such  $\bar{y}$ , the two lines  $IRB$  and  $IRS$  coincide and the asset prices for constrained efficient outcomes are determined by either of them, that is, by requiring equality in condition (38) or (39) with  $y = \bar{y}$ .

(ii.d)  $i > \frac{r+\delta}{1-\delta}$ . See right panel on bottom row in Figure 8. Here  $y^*$  is not implementable. Again, by Lemma 4, the constrained-efficient outcome has  $\ell_s(y) = \ell_b(y)$  and this equality determines the output level  $\bar{y}$ . This also implies that  $\phi = 0$  for the constrained efficient outcome. Moreover, the bubble for the asset in the constrained efficient outcome has  $\ell = \ell_s(\bar{y})$ .  $\square$

## Appendix B. Endogenous resalability constraints

In section 6, we consider a model of resalability constraint in which the cost to recognize assets of types in the interval  $[0, x]$  is  $C(x)$ . It is assumed there that  $C(x) = 0$  for  $x \in [0, \theta]$  and  $C(x) = \infty$  otherwise, and hence the choice of  $x$  is exogenously given. Here we endogenize this choice by assuming that  $C(x) = 0$  for all  $x \in [0, \theta_0]$  for some  $\theta_0 > 0$ , and  $C(x) < \infty$ ,  $C'(x) > 0$ , and  $C''(x) > 0$  for all  $x > \theta_0$ . Therefore, sellers can invest in a costly technology in order to be able to authenticate assets beyond the measure,  $\theta_0$ , that are perfectly recognizable. Again, as in the main text, here we assume that the composition of the buyer's portfolio is common knowledge within the pair. The next lemma derives necessary and sufficient conditions for a constrained-efficient outcome.

**Lemma 5** *There exist  $\bar{\tau}_a \leq \bar{\theta}A$  and  $\bar{\tau}_m \leq 1$  such that  $(\bar{\theta}, \bar{y}, \bar{\tau}_a, \bar{\tau}_m, q, \phi)$  is a constrained-efficient outcome if and only if  $(\bar{\theta}, \bar{y}, q, \phi)$  solves  $\max_{(\theta, y)} \sigma [u(y) - v(y)] - C(\theta)$  subject to  $\ell \equiv q - q^* \geq 0$ ,  $\phi \geq 0$ ,*

$$\sigma \left[ \phi + (1 - \delta)\theta A \ell + \frac{1+r}{\delta+r}\theta A - v(y) \right] - C(\theta) \geq 0 \quad (41)$$

$$-i\phi - (r + \delta)\theta A \ell + \sigma [u(y) - v(y)] - C(\theta) \geq 0. \quad (42)$$

Inequality (41) is a participation constraint for sellers. It states that the expected surplus of sellers in the DM must be greater than the cost they incur in order to recognize a fraction,  $\theta$ , of all assets. Inequality (42) is analogous to a participation constraint for buyers. If buyers receive the whole match surplus net of sellers' information cost, then this expected surplus must be greater than the cost of holding the bubble on Lucas trees and the cost of holding fiat money for buyers to participate in the CM. Constrained-efficient outcomes are characterized in the following proposition.

**Proposition 5 (Information and liquidity)** *Suppose the money supply is growing at the gross growth rate  $\gamma > \beta$ .*

(i) *If  $\theta_0 A \geq A_2$ , then a constrained-efficient outcome has  $y = y^*$  and  $\theta \leq \theta_0$ .*

(ii) *Suppose that  $\theta_0 A < A_2$ .*

(ii.a) *If  $i \leq i^*(\theta_0 A)$ , then a constrained-efficient outcome has  $y = y^*$ ,  $\theta \leq \theta_0$ , with the asset prices determined in the same as in Proposition 4 (ii.a) by replacing  $A$  with  $\theta_0 A$ .*

(ii.b) *If  $i \in \left( i^*(\theta_0 A), \frac{r+\delta}{1-\delta} \right)$ , then the constrained-efficient outcome has  $\ell = 0$ ,  $\phi > 0$ , and the optimal  $(\bar{y}, \bar{\theta})$  are determined as follows: Let  $z(\theta)$  solves  $v(z) - \frac{\sigma}{i}[u(z) - v(z)] = \frac{1+r}{\sigma+r}\theta A - \frac{i+\sigma}{i\sigma}C(\theta)$ ;  $\bar{y} = z(\bar{\theta})$  with  $\bar{\theta}$  satisfying  $\sigma[u'(z(\bar{\theta})) - v'(z(\bar{\theta}))]z'(\bar{\theta}) = C'(\bar{\theta})$ .*

(ii.c) If  $i > \frac{r+\delta}{1-\delta}$ , then the constrained-efficient outcome has  $\ell > 0$ ,  $\phi = 0$ , and the optimal  $(\bar{y}, \bar{\theta})$  are determined as follows: Let  $w(\theta)$  solves  $\frac{1}{1-\delta}v(w) - \frac{\sigma}{r+\delta}[u(w) - v(w)] = \frac{1+r}{(1-\delta)(r+\delta)}\theta A - \left[\frac{1}{\sigma(1-\delta)} + \frac{1}{r+\delta}\right]C(\theta)$ ;  $\bar{y} = w(\bar{\theta})$  with  $\bar{\theta}$  satisfying  $\sigma[u'(w(\bar{\theta})) - v'(w(\bar{\theta}))]w'(\bar{\theta}) = C'(\bar{\theta})$ .

**Proof of Proposition 5.** A level of DM output  $y$  and a fraction  $\theta$  of recognizable assets is feasible if there exists a pair,  $(\ell, \phi)$ , such that:

$$\phi + (1 - \delta)\theta A \ell \geq v(y) - \left(\frac{1+r}{\delta+r}\right)\theta A + \frac{1}{\sigma}C(\theta) \quad (43)$$

$$i\phi + (r + \delta)\theta A \ell \leq \sigma[u(y) - v(y)] - C(\theta). \quad (44)$$

For each output level  $y$ , let  $IRS$  denote the line in the  $(\ell, \phi)$ -space corresponding to (43) at equality, and let  $IRB$  denote the line corresponding to (44) at equality. Define the following intercepts of  $IRS$  and  $IRB$ :

$$\begin{aligned} \phi_s(y, \theta) &= v(y) - \left(\frac{1+r}{\delta+r}\right)\theta A + \frac{1}{\sigma}C(\theta) \\ \ell_s(y, \theta) &= \frac{v(y) - \left(\frac{1+r}{\delta+r}\right)\theta A + \frac{1}{\sigma}C(\theta)}{(1-\delta)\theta A} \\ \phi_b(y, \theta) &= \frac{\sigma[u(y) - v(y)] - C(\theta)}{i} \\ \ell_b(y, \theta) &= \frac{\sigma[u(y) - v(y)] - C(\theta)}{(r+\delta)\theta A}. \end{aligned}$$

$\phi_s(y, \theta)$  (resp.,  $\phi_b(y, \theta)$ ) and  $\ell_s(y, \theta)$  (resp.,  $\ell_b(y, \theta)$ ) are the intercepts of the line  $IRS$  (resp.,  $IRB$ ) with the  $\phi$ -axis and the  $\ell$ -axis, respectively. The pair  $(y, \theta)$  is feasible if and only if  $\phi_s(y, \theta) \leq \phi_b(y, \theta)$  or  $\ell_s(y, \theta) \leq \ell_b(y, \theta)$ .

**Lemma 6** (a) Suppose  $i < \frac{r+\delta}{1-\delta}$ . If  $\ell_s(y, \theta) \leq \ell_b(y, \theta)$ , then  $\phi_s(y, \theta) < \phi_b(y, \theta)$ .

(b) Suppose that  $i = \frac{r+\delta}{1-\delta}$ .  $\phi_s(y, \theta) \leq \phi_b(y, \theta)$  if and only if  $\ell_s(y, \theta) \leq \ell_b(y, \theta)$ .

(c) Suppose that  $i > \frac{r+\delta}{1-\delta}$ . If  $\phi_s(y, \theta) \leq \phi_b(y, \theta)$ , then  $\ell_s(y, \theta) < \ell_b(y, \theta)$ .

First notice that if  $(y, \theta)$  is feasible with  $\theta \leq \theta_0$ , then  $(y, \theta_0)$  is also feasible. Thus,  $(y^*, \theta)$  is feasible for some  $\theta \leq \theta_0$  if and only if  $(y^*, \theta_0)$  is feasible. Now, following the same argument as in Proposition 4,  $(y^*, \theta_0)$  is feasible if and only if  $\theta_0 A \geq A_2$  or  $i \leq i^*(\theta_0 A)$ . This takes care of case (ii.a).

**(ii.b)** Consider now the case  $i \in \left(i^*(\theta_0 A), \frac{r+\delta}{1-\delta}\right)$ . In this case,  $(y, \theta)$  is feasible if and only if  $\phi_s(y, \theta) \leq \phi_b(y, \theta)$ . First we show that at optimal,  $\phi_s(y, \theta) = \phi_b(y, \theta)$ . Because in this case  $(y^*, \theta_0)$  is not feasible,

$(y, \theta)$  is feasible only if  $y < y^*$  or  $\theta > \theta_0$ . If  $\bar{\phi}_s(y, \theta) < \phi_b(y, \theta)$ , then for some  $\varepsilon > 0$ , either  $(y + \varepsilon, \theta)$  or  $(y, \theta - \varepsilon)$  is feasible, a contradiction to optimality of  $(y, \theta)$ .

Define  $z(\theta)$  as the solution  $z$  to  $\phi_s(z, \theta) = \phi_b(z, \theta)$ , that is,

$$v(z) - \frac{\sigma}{i}[u(z) - v(z)] = \frac{1+r}{\sigma+r}\theta A - \frac{i+\sigma}{i\sigma}C(\theta).$$

$z(\theta)$  is well-defined, strictly increasing, and differentiable for  $\theta \in [0, \theta_1^{\max}]$ , where  $\theta_1^{\max}$  satisfies  $\frac{1+r}{\sigma+r}A = \frac{i+\sigma}{i\sigma}C'(\theta)$ . Moreover, for those  $\theta$ 's,

$$z'(\theta) = \frac{\frac{1+r}{\sigma+r}A - \frac{i+\sigma}{i\sigma}C'(\theta)}{v'(z(\theta)) - \frac{\sigma}{i}[u'(z(\theta)) - v'(z(\theta))]} > 0.$$

Because  $v$  is convex and  $u - v$  is concave with  $v(0) = u(0) - v(0)$  and  $\frac{1+r}{\sigma+r}\theta A - \frac{i+\sigma}{i\sigma}C(\theta) > 0$  for those  $\theta$ 's,  $v'(z(\theta)) - \frac{\sigma}{i}[u'(z(\theta)) - v'(z(\theta))] > 0$ . Thus, the welfare maximization problem becomes

$$\max_{\theta \in [0, \theta_1^{\max}]} \sigma[u(z(\theta)) - v(z(\theta))] - C(\theta).$$

Its solution  $\bar{\theta}$  is completely determined by the FOC:  $\sigma[u'(z(\bar{\theta})) - v'(z(\bar{\theta}))]z'(\bar{\theta}) = C'(\bar{\theta})$ .

**(ii.c)** Consider now (ii.c) and assume  $i > \frac{r+\delta}{1-\delta}$ . Then,  $(y, \theta)$  is feasible if and only if  $\ell_s(y, \theta) \leq \ell_b(y, \theta)$ . As in (ii.b), it can be shown that at optimal,  $\ell_s(y, \theta) = \ell_b(y, \theta)$ .

Define  $w(\theta)$  as the solution  $w$  to  $\ell_s(w, \theta) = \ell_b(w, \theta)$ , that is,

$$\frac{1}{1-\delta}v(w) - \frac{\sigma}{r+\delta}[u(w) - v(w)] = \frac{1+r}{(1-\delta)(r+\delta)}\theta A - \left[ \frac{1}{\sigma(1-\delta)} + \frac{1}{r+\delta} \right] C(\theta).$$

$w(\theta)$  is well-defined, strictly increasing, and differentiable for  $\theta \in [0, \theta_2^{\max}]$ , where  $\theta_2^{\max}$  satisfies  $\frac{1+r}{(1-\delta)(r+\delta)}A = \left[ \frac{1}{\sigma(1-\delta)} + \frac{1}{r+\delta} \right] C'(\theta)$ . Moreover, using similar arguments in (ii.b), it is easy to show that for those  $\theta$ 's,

$$w'(\theta) = \frac{\frac{1+r}{(1-\delta)(r+\delta)}A - \left[ \frac{1}{\sigma(1-\delta)} + \frac{1}{r+\delta} \right] C'(\theta)}{\frac{1}{1-\delta}v'(w(\theta)) - \frac{\sigma}{r+\delta}[u'(w(\theta)) - v'(w(\theta))]} > 0.$$

Thus, the welfare maximization problem becomes

$$\max_{\theta \in [0, \theta_2^{\max}]} \sigma[u(w(\theta)) - v(w(\theta))] - C(\theta).$$

Its solution  $\bar{\theta}$  is completely determined by the FOC:  $\sigma[u'(w(\bar{\theta})) - v'(w(\bar{\theta}))]w'(\bar{\theta}) = C'(\bar{\theta})$ .  $\square$