Dynamic Indeterminacy and Welfare in Credit Economies*

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Abstract

We characterize the equilibrium set and constrained-efficient allocations of a pure credit economy with limited commitment under both pairwise and centralized meetings. We show that the set of equilibria derived under "not-too-tight" solvency constraints (e.g., Alvarez and Jermann, 2000; Gu et al., 2013b) is of measure zero in the whole set of Perfect Bayesian Equilibria. There exist a continuum of endogenous credit cycles of any periodicity and a continuum of sunspot equilibria, irrespective of the assumed trading mechanism. Moreover, any allocation of the pure monetary is an allocation of the pure credit economy but the reverse is not true. On the normative side, we establish conditions under which the second welfare theorem of Alvarez and Jermann (2000) fails to apply, i.e., constrained-efficient allocations cannot be implemented with "not-too-tight" solvency constraints.

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1 Introduction

The inability of individuals to commit to honor their future obligations is a key friction of decentralized economies that jeopardizes the Arrow-Debreu apparatus based on promises to deliver goods at different dates and in different states. Economies with limited commitment are the realm of monetary theory. In pure currency economies—such as the one described in Lagos and Wright (2005), among many others—anonymity and lack of commitment make credit infeasible so that all trades are *quid pro quo* and are mediated with money. It has been argued that pure currency economies have become less relevant as technological advances in record-keeping technologies facilitate the use of credit. Yet, monitoring technologies do not make individuals entirely trustworthy, just like they do not make cashless (pure credit) economies frictionless. Hence, in this paper we investigate the full set of equilibria and constrained-efficient allocations of a cashless economy taking seriously the limited commitment friction.[1]

There are two recent contributions, one normative and one positive, that shed some light on non-monetary economies with limited commitment. On the normative side, Alvarez and Jermann (2000), AJ thereafter, establish a second welfare theorem for a pure exchange, one-good economy where agents are subject to endowment shocks and have limited commitment—a special case of the environment in Kehoe and Levine (1993), KL thereafter. They prove that constrained-efficient allocations can be implemented by competitive trades subject to "not-too-tight" solvency constraints. These constraints specify that in every period agents can issue the maximum amount of debt that is incentive-compatible with no default, thereby allowing as much risk sharing as possible. From a positive perspective Gu et al. (2013b), G2MW thereafter, study a pure credit economy subject to the same "not-too-tight" solvency constraints and show the possibility of endogenous credit cycles.[2] The conditions for such cycles, however, are much more stringent than the ones in pure monetary economies.[3]

The objective of this paper is to revisit these two key insights—the implementation of constrained-efficient allocations and the existence of endogenous credit cycles—in the context of a pure credit economy with limited commitment. Our main contributions are twofold. On the positive side, we give a complete characterization of the (perfect Bayesian) equilibrium set of a pure credit economy. On the normative side, we characterize constrained-efficient allocations of economies with pairwise meetings and competitive economies

[1] In Wicksell's (1936) words, <<a thorough analysis of this purely imaginary case seems to me to be worth while, for it provides a precise antithesis to the equally imaginay case of a pure cash system, in which credit plays no part whatever.>>

[2] In a related paper Bloise, Reichlin, and Tirelli (2013) prove indeterminacy of competitive equilibrium in sequential economies under "not-too-tight" solvency constraints. While they do not focus on endogenous credit cycles they show that for any value of social welfare in between autarchy and constrained optimality, there exists an equilibrium attaining that value.

[3] Both Lagos and Wright (2003) and Rocheteau and Wright (2013) find that monetary economies can generate endogenous cycles under monotone trading mechanisms, such as buyers-take-all or proportional bargaining solutions. Under the same trading mechanisms, Gu et al. (2013b) do not find any cycle.

with large meetings.

The pure credit economy we consider features random matching—in pairwise meetings or in large groups—and incorporates intertemporal gains from trade that can be exploited with one-period debt contracts. In the absence of public record keeping, the environment corresponds to the New-Monetarist framework of Lagos and Wright (2005) so that one can easily compare allocations in credit and monetary economies. In the presence of a public record keeping technology the environment is mathematically equivalent to the one in G2MW.[4]

We start with a simple mechanism where the borrower in each bilateral match sets the terms of the loan contract unilaterally, which allows us to analyze the economy as a standard infinitely-repeated game with imperfect monitoring. If we impose the AJ "not-too-tight" solvency constraints exogenously—which amounts to restricting strategies and beliefs such that any form of default is punished with permanent autarky—then there is a unique active steady-state equilibrium and no equilibria with endogenous cycles. When we look for all perfect Bayesian equilibria, we find a continuum of steady-state equilibria, a continuum of periodic equilibria of any periodicity, and much more. Each equilibrium can be reduced to a sequence of debt limits, where the debt limit in a period specifies the amount that agents can be trusted to repay. Moreover, there is a wide variety of outcomes: in some credit cycle equilibria debt limits are binding in all periods, in other equilibria they are never binding, or they bind periodically. These results are robust to the choice of the mechanism to determine the terms of the loan contract—Nash or proportional bargaining, or even Walrasian pricing if agents meet in large groups.

The large multiplicity of credit equilibria captures the basic notion that trust is a self-fulfilling phenomenon. To that extent, and following Mailath and Samuelson (2004, p.9), "we consider multiple equilibria a virtue." But this multiplicity does not imply that everything goes. Fundamentals, including preferences and market structure, do matter for the feasibility of some outcomes. We show that the set of credit-cycle equilibria expands as trading frictions are reduced, agents are more patient, and borrowers have more bargaining power.

We also show that for a given trading mechanism the set of outcomes of a pure monetary economy (with fiat money but no record keeping) is a strict subset of the outcomes of a pure credit economy (with record keeping but no fiat money). So the dynamic allocations of monetary economies (e.g., cycles, chaos...) also exist in credit environments. The reverse is not true. There are outcomes of the pure credit economy that cannot be sustained as outcomes of the pure monetary economy. For instance, there are equilibria where credit and output shut down periodically—which is ruled out by backward-induction in monetary economies.

[4] As we discuss later in details, there are differences regarding the timing of production that are inconsequential.

In order to understand why the equilibrium set for credit economies is so vastly larger than the one found in G2MW (and related papers) it is worth recalling that the AJ "not-too-tight" solvency constraints were meant to provide a way to decentralize constrained-efficient allocations in an economy with limited commitment. Such constraints are not warranted for positive analysis. We avoid arbitrary restrictions on the set of equilibrium outcomes by working with simple strategies that punish both default and excessive lending (i.e., lending in excess to the amount that is deemed trustworthy along the equilibrium path). Any failure to repay a loan that is no greater than the debt limit in the current period triggers permanent autarky for the borrower. However, if a lender agrees on a loan larger than the debt limit, then the borrower remains trustworthy to future lenders provided that he repays at least the debt limit. We will show that such simple strategies implement the full set of outcomes of perfect Bayesian equilibria (subject to mild restrictions).

In terms of normative analysis we determine the constrained-efficient allocations under different assumptions: we consider cases where agents meet in pairs or in large groups and we vary an agent's temptation to renege on his debt. We show that if agents are matched bilaterally then the constrained-efficient allocation is implemented with take-it-or-leave-if-offers by buyers and "not-too-tight" solvency constraints, which generalizes the AJ welfare theorem to economies with pairwise meetings. If meeting sizes are large and agents are price-takers then we consider two cases depending on preferences. When the temptation to renege is high the AJ Second Welfare Theorem holds and the constrained-efficient allocation corresponds to the highest steady state. In contrast, if the temptation to renege is low, then constrained-efficient allocations are non-stationary and incentive constraints to ensure repayment are slack. In this case there exists a continuum of credit cycles that yield a higher social welfare than those with "not-too-tight" solvency constraints (which includes the highest steady state). Slack participation constraints are socially optimal due to a "pecuniary" externality according to which an increase in debt limits allows for higher contemporaneous trade, which raises prices and lowers gains from trade for borrowers, thereby tightening borrowing constraints in earlier periods.

1.1 Related literature

We adopt an environment similar to the pure currency economy of Lagos and Wright (2005) and Rocheteau and Wright (2005) but we replace currency with a public record-keeping technology, as in Sanches and Williamson (2010, Section 4). The first part of the paper on the characterization of the equilibrium set (Sections 3 and 4) extends the analysis of Sanches-Williamson who focus on steady states and G2MW who focus on cycles. Their equilibrium notion imposes the "not-too-tight" solvency constraints of AJ. We show that such constraints have normative foundations that do not apply to our environment. Instead, we present our model as a repeated game with imperfect monitoring.[5] Relative to the repeated game literature we study both stationary and non-stationary equilibria, including endogenous cycles and sunspot equilibria, we consider various trading mechanisms, including ultimatum games, axiomatic bargaining solutions (Kalai and Nash), Walrasian pricing, and we conduct a normative analysis to determine constrained-efficient allocations. Our methods to characterize equilibrium outcomes (Sections 3 and 5) are related but different to the ones used by Abreu (1988) and Abreu, Pierce, and Stacchetti (1990).

Kocherlakota (1998) showed that the set of implementable outcomes of monetary economies is a subset of the implementable outcomes of pure credit economies. In contrast to Kocherlakota we take the trading mechanism as given and we do not restrict outcomes to stationary ones. Hellwig and Lorenzoni (2009) study an environment similar to Alvarez and Jermann (2000) and show that the set of equilibrium allocations with self-enforcing private debt is equivalent to the allocations that are sustained with money. Similarly, Berentsen and Waller (2011) show the equivalence between allocations in an economy with outside money (government bonds) and economy with inside money (private bonds) in a variant of the Lagos-Wright model.

Our paper is part of the literature on limited commitment in macroeconomics. Seminal contributions on risk sharing in endowment economies where agents lack commitment include Kehoe and Levine (1993), Kocherlakota (1996), and Alvarez and Jermann (2000).[6] Kocherlakota (1996) adopts a mechanism design approach in a two-agent economy with a single good. Our Section 5.1 on constrained-efficient allocations under pairwise meetings is related with some key differences: we study a two-good production economy where a continuum of agents search for new partners every period and we select the allocation that maximizes a social welfare criterion under quasi-linear preferences. Gu et al. (2013a, Section 7) has a similar environment but solves for the contract curve. In our Section 5.2 we study constrained-efficient allocations under large meetings and price taking, as in KL or AJ. KL (Section 7) conjectured that punishments based on partial exclusion might allow the implementation of socially desirable allocations.[7] This conjecture is verified in our economy with the caveat that the extent of exclusion has to vary over time. Our normative results are also related to the Second Welfare Theorem in AJ according to which constrained-efficient allocations can be implemented with "not-too-tight" solvency constraints.[8] We will provide a necessary and sufficient

[5] Repeated games where agents are matched bilaterally and at random and change trading partners over time are studied in Kandori (1992) and Ellison (1994). A thorough review of the literature is provided by Mailath and Samuelson (2004).

[6] While our economy is a production economy similar to the one studied in monetary theory it can be easily reinterpreted as an endowment economy along the lines of Rocheteau, Rupert, and Wright (2008).

[7] Similalry, Azariadis and Kass (2013) relaxed the assumption of permanent autarky and assume that agents are only temporarily excluded from credit markets. G2MWa,b allow for partial monitoring which is formally equivalent to partial exclusion, except that the parameter governing the monitoring intensity, π, is time-invariant. Similarly, Kocherlakota and Wallace (1998) consider the case of an imperfect record-keeping technology where the public record of individual transactions is updated after a probabilistic lag.

[8] Early work studying individual bankruptcy and sovereign default (e.g. Eaton and Gersovitz, 1981) were among the first to formalize the notion of endogenous credit constraints to prevent borrowers from defaulting.

condition under which this theorem applies to our environment. When the AJ Second Welfare Theorem fails to apply the constrained-efficient allocation is non-stationary and it is such that borrowers' participation constraints are slack over an infinite number of periods. Finally, in Section 5.3 we also characterize the welfare-maximizing perfect Bayesian equilibria under arbitrary trading mechanisms.

2 Description of the game

Time is discrete and starts with period 0. Each date has two stages. The first stage will be referred to as DM (decentralized market) while the second stage will be referred to as CM (centralized market). There is a single, perishable good at each stage and the CM good will be taken as the numéraire. There is a continuum of agents of measure two divided evenly into a subset of buyers, $\mathbb{B}$, and a subset of sellers, $\mathbb{S}$.[9] The labels "buyer" and "seller" refer to agents' roles in the DM: only the sellers can produce the DM good (and hence will be lenders) and only the buyers desire DM goods (and hence will be borrowers). In the DM a fraction $\alpha \in (0,1]$ of buyers meet with sellers in pairs. (We consider a version of the model with large meetings later.) The CM will be the place where agents settle debts.

Preferences are additively separable over dates and stages. The DM utility of a seller who produces $y \in \mathbb{R}_+$ is $-v(y)$, while that of a buyer who consumes y is $u(y)$, where $v(0) = u(0) = 0$, v and u are strictly increasing and differentiable with v convex and u strictly concave, and $u'(0) = +\infty > v'(0) = 0$. Moreover, there exists $\tilde{y} > 0$ such that $v(\tilde{y}) = u(\tilde{y})$. We denote $y^* = \arg\max[u(y) - v(y)] > 0$ the quantity that maximizes a match surplus. The utility of consuming $z \in \mathbb{R}$ units of the numéraire good is z, where $z < 0$ is interpreted as production.[10] Agents' common discount factor across periods is $\beta = 1/(1+r) \in (0,1)$.

With no loss in generality we restrict our attention to intra-period loans issued in the DM and repaid in the subsequent CM.[11] The terms of the loan contracts are determined according to a simple protocol whereby buyers make take-it-or-leave-it offers to sellers. We describe alternative mechanisms later in the paper. Agents cannot commit to future actions. Therefore, repayment of loans in the CM has to be self-enforcing.

There is a technology allowing loan contracts in the DM and repayments in the CM to be publicly

[9] The assumption of ex-ante heterogeneity among agents is borrowed from Rocheteau and Wright (2005). Alternatively, one could assume that an agent's role in the DM is determined at random in every period without affecting any of our results.

[10] KL and AJ consider pure exchange economies. One could reinterpret our economy as an endowment economy as follows. Suppose that sellers receive an endowment $\bar{y}$ in the DM and $\bar{z}$ in the CM. Buyers have no endowment in the DM but an endowment $\bar{z}$ in the CM. The DM utility of the seller is $w(c)$ where w is a concave function with $w'(\bar{y}) = 0$. Hence, the opportunity cost to the seller of giving up y units of consumption is $v(y) = w(\bar{y}) - w(\bar{y} - y)$. It can be checked that v is convex with $v(0) = 0$ and $v'(y) = -w'(\bar{y} - y)$.

[11] Under linear payoffs in the CM one-period debt contracts are optimal, i.e., agents have no incentives to smooth the repayment of debt across multiple periods. This assumption will facilitate the comparison with pure monetary economies of the type studied in Lagos and Wright (2005).

recorded. The entry in the public record for each loan is a triple, (ℓ, x, i), composed of the size of the loan negotiated in the DM in terms of the numéraire good, $\ell \in \mathbb{R}_+$, the amount repaid in the CM, $x \in \mathbb{R}_+$, and the identity of the buyer, $i \in \mathbb{B}$. If no credit is issued in a pairwise meeting, or if i was unmatched, the entry in the public record is $(0, 0, i)$. The record is updated at the end of each period t as follows:

$$\rho^{i,t+1} = \rho^{i,t} \circ (\ell_t, x_t, i), \tag{1}$$

where $\rho^{i,0} = (\ell_0, x_0, i)$. The list of records for all buyers, $\rho^t = \langle \rho^{i,t} : i \in \mathbb{B} \rangle$, is public information to all agents.[12] Agents have private information about their trading histories that are not recorded; in particular, if $\rho^{i,t} = (0, 0, i)$, then agents other than i do not know whether i was matched but his offer got rejected (in that case, the offer made is not observed either) or was unmatched. However, as discussed later, this private information plays no role in our construction of equilibria.

3 Equilibria

For each buyer $i \in \mathbb{B}$, a strategy, s^i, consists of two functions $s^i_t = (s^i_{t,1}, s^i_{t,2})$ at each period t, conditional on being matched: $s^i_{t,1}$ maps his private trading history, $h^{i,t}$, and public records of other buyers, $\rho^{-i,t}$, to an offer to the seller, (y_t, ℓ_t); $s^i_{t,2}$ maps $((h^{i,t}, \rho^{-i,t}), (y_t, \ell_t))$, together with the seller's response, to his CM repayment, x_t. For each seller $j \in \mathbb{S}$, a strategy, s^j, consists of one function at each period t, conditional on being matched with buyer i: s^j_t maps the seller's private trading history, $h^{j,t}$, the buyer's identity and public records, $(i, \rho^{i,t}, \rho^{-i,t})$, and his last offer, (y_t, ℓ_t), to a response, *yes* or *no*. We restrict our attention to (weak) perfect Bayesian equilibria (equivalently, weak sequential equilibria) satisfying the following conditions:

(A1) **Public strategies.** In any DM meeting the strategies only depend on histories that are common knowledge in the match, including the buyer's public trading history, his offer and the seller's response in the current match, but not on private histories (nor the public records of other buyers).[13]

(A2) **Symmetry.** All buyers adopt the same strategy, s^b, and all sellers adopt the same strategy, s^s. Moreover, the buyer's offer strategy, $s^b_{t,1}$, is constant over all public trading histories of the buyer that are consistent with equilibrium behavior, in particular, equilibrium offers at date-t are independent of matching histories.

[12] We could make alternative assumptions regarding what is recorded in a match. For instance, the technology could also record the output level, y, together with the pomises made by the buyer, i.e., $\rho^i = (y, \ell, x, i)$. Not surprisingly, this would expand the set of equilibrium outcomes. Alternatively, the promised repayment of the buyer might not be recorded, in which case ρ^i is simply (x, i). Maybe more surprisingly, this would also expand the equilibrium set by allowing allocations where y is larger than y^*. Moreover, we could assume that the seller only observes the record of the buyer he is matched with, ρ^i, without affecting our results.

[13] For a formal definition of public strategies see Definition 7.1.1 in Mailath and Samuelson (2006). We extend this definition to our set-up by allowing strategies to depend not only on the buyer's public history, $\rho^{i,t}$, but also histories that are common knowledge in a given match.

(A3) **Threshold rule for repayments.** For each buyer i and each date t following any history, there exists a number, d_t, such that d_t is weakly larger than the equilibrium loan amount at date t, and $s^b_{t,2}(\rho^{i,t}, (y_t, \ell_t), yes) = \ell_t$ if $\ell_t \leq d_t$ and if $\rho^{i,t}$ is consistent with equilibrium behavior.

We call a (weak) perfect Bayesian equilibrium, (s^b, s^s), satisfying conditions (A1)-(A3) above a *credit equilibrium*. A few remarks are in order about these conditions. Our record keeping technology does not record all actions taken by the agents. Agents have private information about the number of matches they had, quantities they consumed, or offers that were rejected. Because of this private information using perfect Bayesian equilibrium (PBE) as the solution concept is both standard and necessary.[14] Alternatively, one may assume that all actions are observable, and PBE is reduced to subgame perfection. Although we prefer our environment, which is closer to the existing literature on monetary economics, our multiplicity result does not rely on the presence of private information. In fact, because of our focus on public strategies, (A1), any PBE we construct is also a subgame-perfect equilibrium (SPE) if all actions were observable.[15] However, agents' belief about how other agents will respond to deviations do matter but they are pinned down by equilibrium strategies.

Conditions (A1) and (A2) imply that, for any credit equilibrium, its outcomes are characterized by $\{(y_t, \ell_t)\}_{t=0}^{+\infty}$, the sequence of equilibrium offers made by buyers. Moreover, (A3) implies that $x_t = \ell_t$ for each t, and hence the sequence $\{(y_t, \ell_t)\}_{t=0}^{+\infty}$ also determines the equilibrium allocation. Without (A1), equilibrium offers may depend on the buyer's past matching histories.[16] Condition (A3) is not vacuous either. It restricts sellers to believe that buyers will repay their debt when observing a deviating offer with obligations smaller than those in equilibrium.[17] This restriction will rule out inefficiently large trades. As we will see later, taken together the restrictions (A1)-(A3) will allow us to obtain a simple representation of credit equilibria with solvency constraints added to the bargaining problem.

Let $\{(y_t, \ell_t)\}_{t=0}^{+\infty}$ be a sequence of equilibrium offers. Along the equilibrium path the lifetime expected discounted utility of a buyer at the beginning of period t is

$$V^b_t = \sum_{s=0}^{\infty} \beta^s \alpha[u(y_{t+s}) - \ell_{t+s}]. \tag{2}$$

In each period $t+s$ the buyer is matched with a seller with probability α in which case the buyer asks for

[14] A PBE consists of a list of strategies, one for each player, and, at each information set, a belief system that specifies for each player a distribution over all possible histories consistent with their information. It requires sequential rationality and the beliefs to be consistent with the Bayes rule whenever possible.

[15] In such an equilibrium sellers' beliefs about buyers' private information are irrelevant for their decisions to accept or reject offers. Hence, actions that correspond to agents' private trading histories would not matter even if they were publicly observable.

[16] Obviously, when $\alpha = 1$, the matching-history-independence element in (A1) is vacuous. However, when $\alpha < 1$, it would be difficult to fully characterize all equilibrium outcomes without (A1) but it certainly adds many more equilibria.

[17] Without this restriction one could sustain equilibria in which $y_t > y^*$ for some t; to do so, one can adopt a strategy that triggers a permanent autarky for the buyer if his offer ℓ_t is smaller than the equilibrium one.

y_{t+s} units of DM output in exchange for a repayment of ℓ_{t+s} units of the numéraire in the following CM and the seller agrees. In any equilibrium $-\ell_t + \beta V_{t+1}^b \geq 0$, which simply says that a buyer must be better off repaying his debt and going along with the equilibrium rather than defaulting on his debt and offering no-trade in all future matches, $(y_{t+s}, \ell_{t+s}) = (0, 0)$ for all $s > 0$. By a similar reasoning the lifetime expected utility of a seller along the equilibrium path is

$$V_t^s = \sum_{s=0}^{\infty} \beta^s \alpha[-v(y_{t+s}) - \ell_{t+s}]. \tag{3}$$

The seller's participation constraint in the DM requires $-v(y_t) + \ell_t \geq 0$ since a seller can reject a trade without fear of retribution. Given that buyers set the terms of trade unilaterally, and the output level is not part of the record ρ^i, this participation constraint holds at equality. Our first proposition builds on these observations to characterize outcomes of credit equilibria.

Proposition 1 *A sequence, $\{(y_t, x_t, \ell_t)\}_{t=0}^{\infty}$, is a credit equilibrium outcome if and only if, for each $t =$ $0, 1, ...,$*

$$\ell_t \leq \sum_{s=1}^{\infty} \beta^s \alpha[u(y_{t+s}) - \ell_{t+s}] \tag{4}$$

$$\ell_t = x_t = v(y_t) \leq v(y^*). \tag{5}$$

As mentioned earlier, a sequence of equilibrium offers, $\{(y_t, \ell_t)\}_{t=0}^{+\infty}$, also determines the sequence of allocations, $\{(y_t, x_t)\}_{t=0}^{+\infty}$, with $x_t = \ell_t$ for each t, and hence, Proposition 1 also gives a characterization of allocations that can be sustained in a credit equilibrium. In this sense, condition (4) is analogous to the participation constraint (IR) in KL, and the participation constraint in Proposition 2.1 in Kocherlakota (1996). However, while in KL the IR constraint is assumed from the outset as a primitive condition, condition (4) is derived as an equilibrium condition in our framework. It follows directly from (2) and the incentive constraint $-\ell_t + \beta V_{t+1}^b \geq 0$.

The condition (5) is the outcome of the buyer take-it-or-leave-if offer and pairwise Pareto efficiency (which follows from the threshold rule A3).[18] Proposition 1 shows that the conditions (4)-(5) are not only necessary but also sufficient for an equilibrium by constructing a simple strategy profile. This strategy profile relies on punishments—the "penal code" in Abreu's (1988) terminology—for both default and excessive lending.[19] Specifically, buyers can be in two states at the beginning of period t, $\chi_{i,t} \in \{G, A\}$, where G means "good

[18] To derive these conditions formally one has to use of the assumption that y_t is not publicly recorded—only the loan contract is—and the threshold property in (A3). See proof of Proposition 1.

[19] There are different approaches for finding equilibria of repeated games. Abreu, Pearce, and Stacchetti (1990) introduce the idea of self-generating set of equilibrium payoffs while Abreu (1988) introduces the notion of simple strategies. See Mailath and Samuelson (2006, Section 2.5) for a review of these approaches.

standing" and A means "autarky", and each buyer's initial state is $\chi_{i,0} = G$. The law of motion of the buyer i's state following a loan and repayment $(\tilde{\ell}, \tilde{x})$ are given by:

$$\chi_{i,t+1}\left(\tilde{\ell}, \tilde{x}, \chi_{i,t}\right) = \begin{cases} A & \text{if } \tilde{x} < \min(\tilde{\ell}, \ell_t) \text{ or } \chi_{i,t} = A \\ G & \text{otherwise} \end{cases}, \tag{6}$$

where $(\tilde{\ell}, \tilde{x})$ might differ from the loan and repayment along the equilibrium path, $\ell_t = x_t$. In order to remain in good standing, or state G, the buyer must repay his loan, $\tilde{x} \geq \tilde{\ell}$, if the size of the loan is no greater than the equilibrium loan size, $\tilde{\ell} \leq \ell_t$, and he must repay the equilibrium loan size, $\tilde{x} \geq \ell_t$, otherwise.[20] The autarky state, A, is absorbing: once a buyer becomes untrustworthy, he stays untrustworthy forever.[21] Sellers cannot be punished in future periods for accepting a loan larger than ℓ_t since their identity is not recorded. However, they are punished in the current period because buyers are allowed to partially default on loans larger than ℓ_t while keeping their good standing with future lenders.

The strategies, (s^b, s^s), depend on the buyer's state as follows. The seller's strategy, s_t^s, consists in accepting all offers, $(\tilde{y}, \tilde{\ell})$, such that $\upsilon(\tilde{y}) \leq \min\{\tilde{\ell}, \ell_t\}$ provided that the buyer's state is $\chi_{i,t} = G$. The buyer repays $s_{t,2}^b = \min\{\ell_t, \tilde{\ell}\}$ if he is in state G, and he does not repay anything otherwise, $s_{t,2}^b = 0$. These strategies are depicted in Figure 1 where (y_t, ℓ_t) is the offer made by a buyer in state G along the equilibrium path and $(\tilde{y}, \tilde{\ell})$ is any offer. By the one-stage-deviation principle it is then straightforward to show that any $\{(y_t, \ell_t)\}_{t=0}^{\infty}$ that satisfies (4)-(5) is an outcome for the strategy profile (s^b, s^s).

In the following we propose an alternative formulation of a credit equilibrium in terms of solvency constraints imposed to the bargaining problems in the DMs. As in AJ in the context of an economy with competitive trades, a solvency constraint specifies an upper bound—called a *debt limit*—on the quantity of debt an agent can issue, $\ell_t \leq d_t$. According to this formulation, the buyer in a DM match sets the terms of the loan contract so as to maximize his surplus, $u(y_t) - \ell_t$, subject to the seller's participation constraint and the solvency (or borrowing) constraint, $\ell_t \leq d_t$, i.e.,

$$\max_{y_t, \ell_t} \{u(y_t) - \ell_t\} \quad \text{s.t.} \quad -\upsilon(y_t) + \ell_t \geq 0 \;\text{ and }\; \ell_t \leq d_t. \tag{7}$$

The solution to (7) is $\ell_t = \upsilon(y_t)$ where

$$y_t = z(d_t) \equiv \min\{y^*, \upsilon^{-1}(d_t)\}. \tag{8}$$

[20] Note that the buyer can remain in state G even if he does not pay his debt in full, and hence default is with respect to the common belief that buyers repay up to the size of the equilibrium loan. Also, notice that there are alternative strategy profiles that deliver the same equilibrium outcome. For instance, an alternative automaton is such that the transition to state A only occurs if $\tilde{x} < \tilde{\ell} \leq \ell_t$. If a loan such that $\tilde{\ell} > \ell_t$ is accepted, then the buyer can default without fear of retribution.

[21] While no player has an incentive to deviate unilaterally along a subgame perfect equilibrium, at some heuristic level it seems that two players may want to "renegotiate" the punishment and coordinate on some preferred outcome in the Pareto sense. In the context of our credit equilibrium, the seller might want to forgive, and trust, a buyer who defaulted in the past instead of punishing him with no trade. This idea was formalized by Farrell and Maskin (1989), among others, with the refinement concept of weakly renegotiation-proof (WRP) equilibrium, defined as one where any two continuation payoffs are not Pareto-rankable. From our viewpoint, WRP eliminates the possibility of interesting coordination failures and the possibility of belief-driven credit cycles.

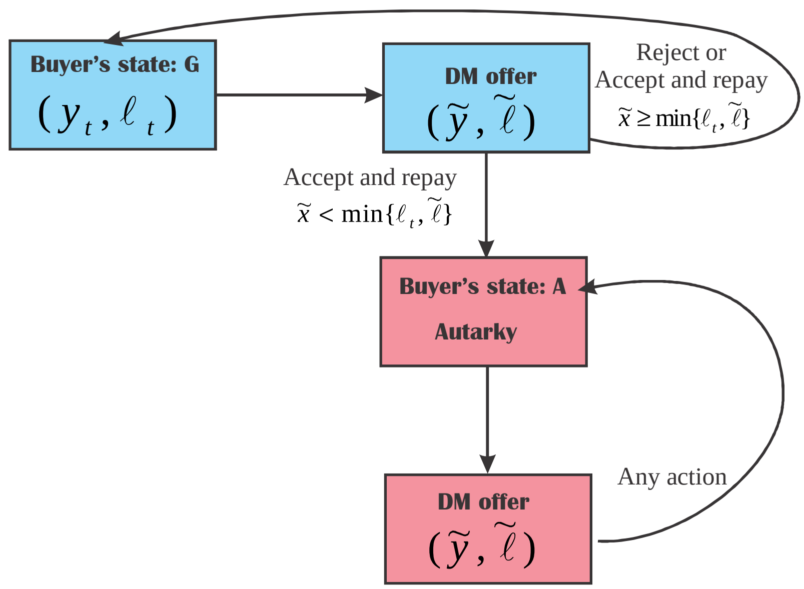

Figure 1: Automaton representation of the buyer's strategy

The solvency constraint is reminiscent to the feasibility constraint in monetary models (e.g., Lagos and Wright, 2005) according to which buyers in bilateral matches cannot spend more than their real balances.

We say that a sequence of debt limits, $\{d_t\}_{t=0}^{\infty}$, is consistent with a credit equilibrium outcome, $\{(y_t, x_t, \ell_t)\}_{t=0}^{\infty}$, if (y_t, ℓ_t) is a solution to the bargaining problem, (7), given d_t for all $t \in \mathbb{N}_0$, and the buyer's CM strategy consists in repaying his debt up to d_t provided that his past public histories (up to period $t-1$) are consistent with equilibrium behavior. It is easy to check from the proof of Proposition 1 that any credit equilibrium outcome, $\{(y_t, x_t, \ell_t)\}_{t=0}^{\infty}$, is consistent with a sequence of debt limits, $\{d_t\}_{t=0}^{\infty}$, such that $d_t = \ell_t$ for all $t \in \mathbb{N}_0$. But the same equilibrium outcome may be implementable by multiple debt limits if (9) is slack and $y_t = y^*$. The buyers' repayment strategy is obtained from an automaton analogous to the one in Figure 1 where ℓ_t is replaced with d_t for all t. The following corollary shows that a credit equilibrium reduces to a sequence of debt limits, $\{d_t\}_{t=0}^{\infty}$, that satisfies a sequence of participation constraints.

Corollary 1 ***(Equilibrium representation with debt limits)*** *A sequence of debt limits, $\{d_t\}_{t=0}^{\infty}$, is consistent with a credit equilibrium outcome if and only if*

$$d_t \leq \sum_{s=1}^{\infty} \beta^s \alpha [u(y_{t+s}) - v(y_{t+s})] \tag{9}$$

$$v(y_t) = \min\{d_t, v(y^*)\}. \tag{10}$$

Corollary 1 gives a complete characterization of equilibrium outcomes using debt limits. Indeed, by (10), y_t is determined by d_t, and hence (9) can be viewed as an inequality that involves $\{d_t\}_{t=0}^{\infty}$ as the only endogenous variables. Without the danger of confusion, we also call a sequence of debt limits, $\{d_t\}_{t=0}^{\infty}$, a credit equilibrium if it satisfies (9) and (10).

While AJ introduces the solvency constraint as a primitive condition, we derive the debt limits endogenously as part of equilibrium strategies. AJ focuses on solvency constraints that are "not-too-tight," meaning that d_t is the largest debt limit that solves the buyer's CM participation constraint, (9), at equality, thereby preventing default while allowing as much trade as possible. A "too-tight" solvency constraint is such that (9) is slack. In contrast to AJ and G2MW, we do not impose the buyer's participation constraint to bind, i.e., the solvency constraint to be "not-too-tight," as such restriction would reduce the equilibrium set dramatically and might eliminate equilibria with good welfare properties. The next Corollary provides a sufficient condition for an equilibrium in recursive form.

Corollary 2 ***(Recursive sufficient condition)*** *Any bounded sequence, $\{d_t\}_{t=0}^{+\infty}$, that satisfies*

$$d_t \leq \beta \left\{ \alpha \left[u(y_{t+1}) - \upsilon(y_{t+1}) \right] + d_{t+1} \right\}, \tag{11}$$

where $\upsilon(y_t) = \min\{d_t, \upsilon(y^*)\}$, is a credit equilibrium.

The left side of (11) is the cost of repaying the current debt limit while the right side of (11) is the benefit that has two components: the expected match surplus of a buyer who has access to credit and his continuation value given by the next-period debt limit. In Figure 2 we represent (11) holding at equality by a red curve. We plot a truncated sequence of debt limits, (d_{T-2}, d_{T-1}, d_T), that solves (11), i.e., (d_{T-2}, d_{T-1}) and (d_{T-1}, d_T) are located to the left of the red curve. Under "not-too-tight" solvency constraints $\{d_t\}$ solves (11) where the weak inequality is replaced with an equality and hence any pair, (d_{t-1}, d_t), is on the red curve. The sequence of inequalities, (11), are sufficient conditions for a credit equilibrium, but they are not necessary. We will provide examples later of credit equilibria that do not satisfy (11).

3.1 Steady-state equilibria

We characterize first steady states where debt limits and DM allocations are constant through time, $(d_t, y_t, \ell_t) = (d, y, \ell)$ for all t. Under such restriction the incentive-compatibility condition, (9), or, equivalently, (11), can be simplified to read:

$$rd \leq \alpha \left\{ u\left[z(d)\right] - \upsilon\left[z(d)\right] \right\}, \tag{12}$$

where z given by (8) indicates the DM level of output as a function of d. The left side of (12) is the flow cost of repaying debt while the right side is the flow benefit from maintaining access to credit. This benefit is equal to the probability of a trading opportunity, α, times the whole match surplus, $u(y) - \upsilon(y)$, where $y = z(d)$.

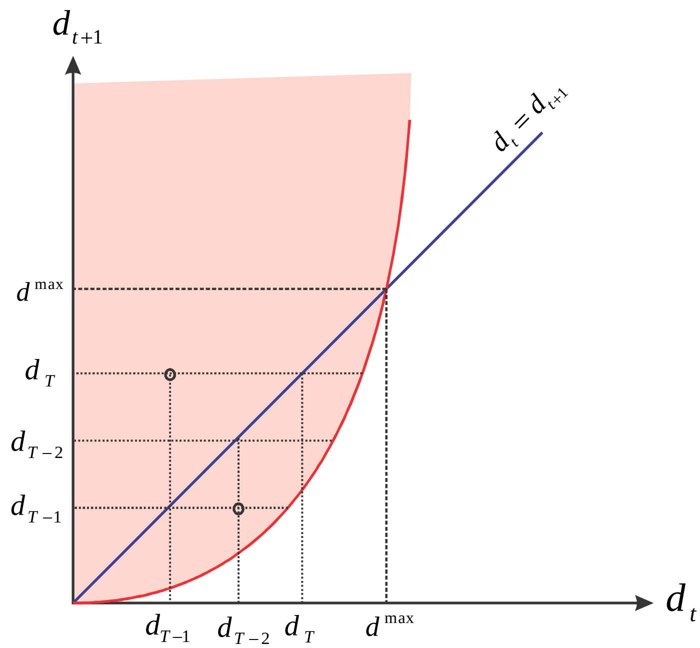

Figure 2: Recursive representation

Let $d^{\max}$ denote the highest value of the debt limit that satisfies (12), i.e., $d^{\max}$ is the unique positive root to $rd^{\max} = \alpha\{u[z(d^{\max})] - v[z(d^{\max})]\}$. It is determined graphically in Figure 3 at the intersection of the left side of (12) that is linear and the right side of (12) that is concave. For all $d < d^{\max}$ the gain from defaulting is less than the cost associated with permanent autarky. The next Proposition shows that any debt limit between $d = 0$ and $d = d^{\max}$ is part of an equilibrium.

Proposition 2 ***(Steady-State Equilibria)*** *There exists a continuum of steady-state, credit equilibria indexed by $d \in [0, d^{\max}]$ with $d^{\max} > 0$.*

The two extreme debt limits, $\{0, d^{\max}\}$, correspond to the two steady-state equilibria under the AJ "not-too-tight" solvency constraints where the gain from defaulting is exactly equal to the cost of permanent autarky. Proposition 2 establishes that any debt limit in between these two extreme values is also part of an equilibrium. The intuition is as follows. For any debt d between 0 and $d^{\max}$ the gain from defaulting is strictly smaller than the cost associated with permanent autarky. So the buyer has incentives to repay such a loan. What about a slightly larger loan? If a buyer offers $\ell > d$ then he only repays d, which is the repayment that keeps him trustworthy to other sellers. As a result any loan such that $\ell > d$ is treated like a promise to repay d and it is acceptable provided that $v(y) \leq d$. Obviously, the borrower would like to convince his current lender that he could repay more than d, but this promise is not credible as his incentive to repay depends on future access to credit.

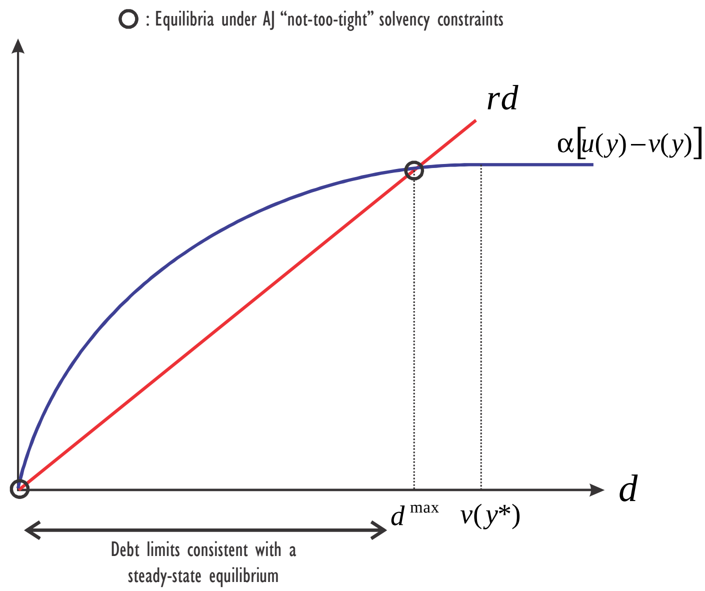

Figure 3: Set of debt limits at steady-state, credit equilibria

3.2 Periodic equilibria

Here we consider deterministic credit cycles where the extent to which buyers are trustworthy to repay their debts changes over time. We start with 2-period cycles, $\{d_0, d_1\}$, where d_0 is the debt limit in even periods and d_1 is the debt limit in odd periods. The incentive-compatibility condition, (9), becomes:

$$rd_t \le \frac{\alpha\left[u(y_{(t+1)\bmod 2}) - v(y_{(t+1)\bmod 2})\right] + \beta\alpha\left[u(y_t) - v(y_t)\right]}{1+\beta}, \quad t \in \{0,1\}, \tag{13}$$

where we used that $\ell_t = v(y_t)$ from (10). The term on the numerator on the right side of (13) is the buyer's expected discounted utility over the 2-period cycle starting in $t+1$. Obviously, the steady-state equilibria described in Proposition 2 are special cases of two-period cycles; indeed, for any $d \in [0, d^{\max}]$, (d, d) satisfy (13). We define, for each $d_0 \in [0, d^{\max}]$,

$$\gamma(d_0) \equiv \max\{d_1 : (d_0, d_1) \text{ satisfies (13) with } t = 1\}, \tag{14}$$

the highest debt limit in odd periods consistent with a debt limit equal to d_0 in even periods. A 2-period-cycle equilibrium, or simply a *2-period cycle*, is a pair (d_0, d_1) that satisfies $d_0 \le \gamma(d_1)$ and $d_1 \le \gamma(d_0)$.

Lemma 1 *The function $\gamma(d)$ is positive, non-decreasing, and concave. Moreover, $d^{\min} \equiv \gamma(0) > 0$, $\gamma(d) > d$ for all $d \in (0, d^{\max})$, and $\gamma(d^{\max}) = d^{\max}$. If $v(y^*) < d^{\max}$, then $\gamma(d) = d^{\max}$ for all $d \in [v(y^*), d^{\max}]$.*

The function γ is represented in the two panels of Figure 4. It is non-decreasing because if the debt limit in even periods increases, then the punishment from defaulting gets larger and, as a consequence, higher debt limits can be sustained in odd periods. So there are complementarities between agents' trustworthiness in odd periods and agents' trustworthiness in even periods. The function $\gamma(d)$ is always positive because even if credit shuts down in even periods, it can be sustained in odd periods by the threat of autarky in both odd and even periods. For a given d_0 we define the set of debt limits in odd periods that are consistent with a 2-period cycle by

$$\Omega(d_0) \equiv \{d_1 : d_0 \leq \gamma(d_1), d_1 \leq \gamma(d_0)\}. \tag{15}$$

In Figure 4 the set of credit cycles is the area between γ and its mirror image with respect to the 45$^{\text{o}}$ line.

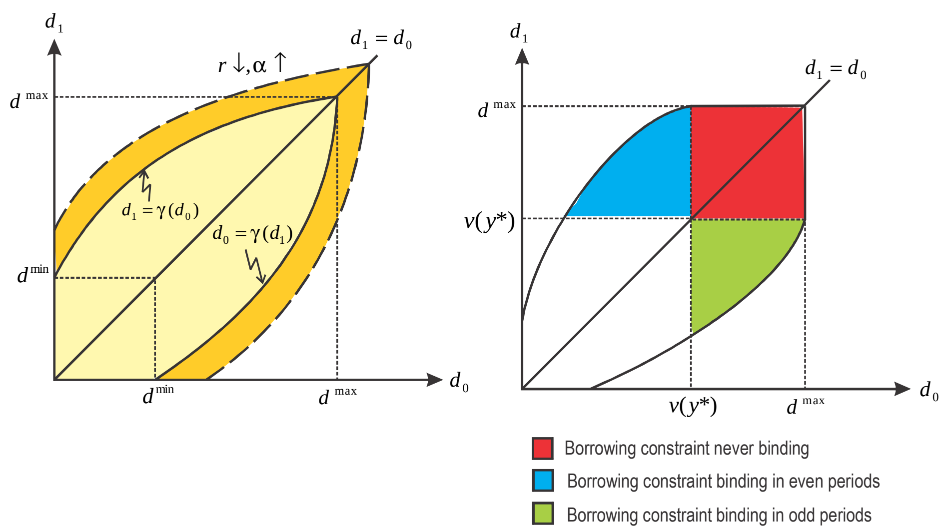

Figure 4: Set of two-period credit cycles

Proposition 3 ***(2-Period Credit Cycles)*** *For all $d_0 \in [0, d^{\max})$ the set of 2-period credit cycles with initial debt limit, d_0, denoted $\Omega(d_0)$, is a nondegenerate interval.*

G2MW restrict attention to equilibria such that $d_0 = \gamma(d_1)$ and $d_1 = \gamma(d_0)$, i.e., at the debt limit borrowers are exactly indifferent between repaying their debt and not repaying it and moving to autarky. Given the monotonicity and concavity of $\gamma(d)$ such equilibria do not occur outside of the 45$^{\text{o}}$ line, i.e., there are no (strict) credit cycles. Indeed, if $d_0 \in (0, d^{\max})$ then the maximum debt limit in odd periods is $d_1 = \gamma(d_0) > d_0$. But given d_1 the maximum debt limit in even periods is $d_0' = \gamma(d_1) > d_1 > d_0$. Following this argument we obtain an increasing sequence, $\{d_0, \gamma(d_0), \gamma(\gamma(d_0)), ...\}$, that converges to $d^{\max}$. In contrast we find a continuum of (strict) two-period cycle equilibria. Moreover, the set of steady-state equilibria is of measure 0 in the set of all 2-period equilibria. Indeed, for any d_0 in the interval $(0, d^{\max})$ there are a continuum of two-period cycles where d_0 is the debt limit in even periods.

The set of credit equilibria described in Proposition 3 contains equilibria where credit dries up periodically. In the left panel of Figure 4 such equilibria correspond to the case where $d_0 = 0 < d_1 < \gamma(0) = d^{\min}$, i.e., even-period IOUs are believed to be worthless while odd-period IOUs are repaid. If a seller extends a loan in an even period, the buyer defaults, in accordance with beliefs, but remains trustworthy in subsequent odd periods. Such outcomes are ruled out by backward induction in pure-currency economies. In contrast a credit economy has IOUs issued at different dates (and by different agents), and hence agents can form different beliefs regarding the terminal value of these different assets.

The result according to which there are a continuum of equilibria does not imply that everything goes. Fundamentals, such as preferences and matching technology, do matter for the outcomes that can emerge. The following corollary investigates how changes in fundamentals affect the equilibrium set.

Corollary 3 ***(Comparative statics)*** *As r decreases or α increases the set of 2-period cycles expands.*

If agents become more patient, i.e., r decreases, then γ shifts upward, as the discounted sum of future utility flows associated with a given allocation increases, and the set of 2-period cycle equilibria expands. The expansion of the equilibrium set is represented by the dark yellow area in the left panel of Figure 4. Similarly, if the frequency of matches, α, increases, then $d^{\max}$ increases as permanent autarky entails a larger opportunity cost, and the set of credit cycles expands.

Corollary 4 ***(Credit tightness over the cycle)***

If $r \geq \alpha \left[u(y^*) - \upsilon(y^*)\right]/\upsilon(y^*)$ then $\ell_t \leq d_t$ binds for both $t \in \{0, 1\}$ in any 2-period cycle.

If $r < \alpha \left[u(y^*) - \upsilon(y^*)\right]/\upsilon(y^*)$, then there are 2-period cycles such that $\ell_t \leq d_t$ is slack for all $t \in \{0, 1\}$, and there are 2-period cycles where $\ell_t \leq d_t$ binds only periodically.

Corollary 4 shows that if agents are sufficiently impatient, as in the left panel of Figure 4, then the debt limit binds and output is inefficiently low in every period for all credit cycles. However, if agents are patient, then there are equilibria where the debt limit binds periodically. Such equilibria are represented by the blue and green areas in the right panel of Figure 4. There are also equilibria where the debt limit fluctuates over time but never binds. These fluctuations, however, are payoff-irrelevant since the allocation is constant and the first best is implemented, $y_0 = y_1 = y^*$. These equilibria are represented by the red square, $[\upsilon(y^*), d^{\max}]^2$, in the right panel of Figure 4.

One can generalize the above arguments to T-period cycles, $\{d_j\}_{j=0}^{T-1}$. The debt limits must solve the following inequalities:

$$d_t \leq \frac{\alpha \sum_{j=1}^{T} \beta^j \left\{ u\left[y_{(t+j) \bmod T}\right] - \upsilon\left[y_{(t+j) \bmod T}\right] \right\}}{1 - \beta^T}, \quad t = 0, ..., T-1 \tag{16}$$

The numerator on the right side of (16) is the expected discounted sum of utility flows over the T-period cycle. Following the same reasoning as above:

Proposition 4 ***(T-Period Credit Cycles)*** *For any $T \geq 2$ and for all $d_0 \in [0, d^{\max})$, the set of T-period credit cycles with initial debt limit, d_0, denoted $\Omega_T(d_0)$, is a bounded, convex, and closed set in $\mathbb{R}^{T-1}$ with positive Lebesgue measure.*

Our environment can lead to cycles of any periodicity, and for a given length of the cycle there are a continuum of equilibria. As an illustration, in Figure 5 we represent the set of 3-period for a given parametrization. One can see from the right panel that there is a non-empty set of 3-period cycles (the pink area) where credit shuts down periodically, once or twice every three periods. Also, for our parametrization the first best is implementable, i.e., there are equilibria with $d_t \geq 1$ for all $t \in \{0, 1, 2\}$. More generally, one can check that the set of T-period cycles is of measure 0 in the set of $(T+1)$-period cycles.

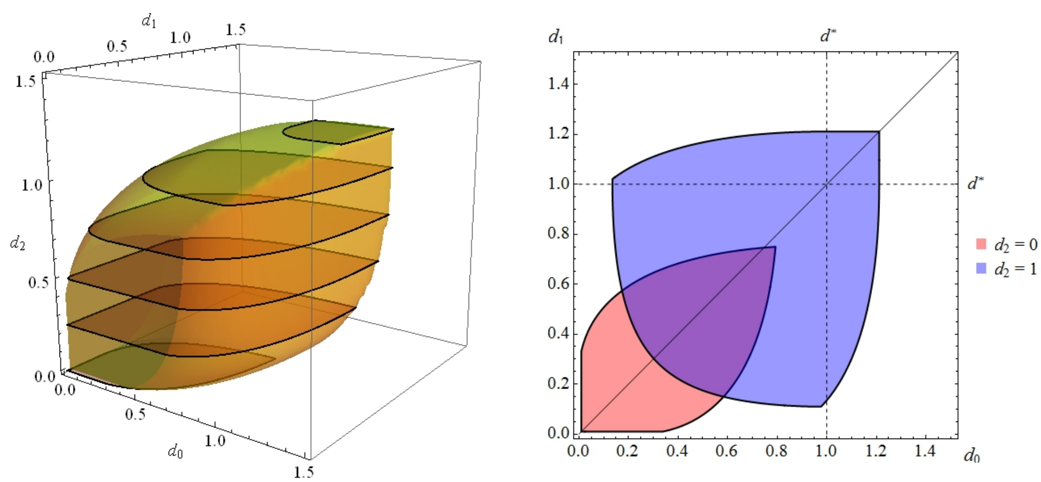

Figure 5: Set of three-period credit cycles: $u(y) = 2\sqrt{y}, v(y) = y, \beta = 0.9, \alpha = 0.25$

3.3 Monetary vs credit equilibria

We now consider the same environment as before but we shut down the record-keeping technology: individual trading histories are private information in a match. Without public memory credit is no longer incentive-feasible as a buyer would find it optimal to renege on his debt. Suppose that all buyers are endowed with $M = 1$ units of fiat money at time $t = 0$. Money is a perfectly divisible, storable, and intrinsically useless object, and its supply is constant over time. The environment is now identical to the one in Lagos and Wright

(2003, 2005).[22] We show in the following that any allocation, $\{(x_t, y_t)\}_{t=0}^{+\infty}$, of a pure monetary economy, where x_t is CM output and y_t is DM output, is also an allocation of a pure credit economy.

The CM price of money in terms of the numéraire good is denoted ϕ_t. The buyer's choice of money holdings in period t is the solution to the following problem:

$$\max_{m \geq 0} \left\{ -\phi_t m + \beta\alpha \left[u(y_{t+1}) - \upsilon(y_{t+1}) \right] + \beta\phi_{t+1} m \right\}, \tag{17}$$

where, from buyers' take-it-or-leave-it offers in the DM, sellers are indifferent between trading and not trading, $\upsilon(y_t) = \phi_t m$. From (17) it costs $\phi_t m$ to the buyer in the CM of period t to accumulate m units of money. In the following DM the buyer can purchase $y_{t+1} = \upsilon^{-1}(\phi_{t+1} m)$ if he happens to be matched with probability α. Otherwise the buyer can resale his units of money at the price ϕ_{t+1} in the CM of period $t+1$. From the first-order condition of (17), $\{\phi_t\}_{t=0}^{+\infty}$ solves the following first-order difference equation,

$$\phi_t = \beta\phi_{t+1} \left[1 + \alpha \frac{u'(y_{t+1}) - \upsilon'(y_{t+1})}{\upsilon'(y_{t+1})} \right]. \tag{18}$$

According to (18) the value of fiat money in period t is equal to the discounted value of money in period $t+1$ augmented with a liquidity term that captures the expected marginal surplus from holding an additional unit of money in a pairwise meeting in the DM. A monetary equilibrium is a bounded sequence, $\{(y_t, x_t, \phi_t)\}_{t=0}^{+\infty}$, that solves (18), with $\upsilon(y_t) = x_t = \min\{\phi_t, \upsilon(y^*)\}$.

Proposition 5 ***(Monetary vs Credit Equilibria)*** *Let $\{(y_t, x_t, \phi_t)\}_{t=0}^{+\infty}$ be a monetary equilibrium of the economy with no record-keeping. Then, $\{(y_t, x_t, \ell_t)\}_{t=0}^{+\infty}$ is a credit equilibrium of the economy with record-keeping, where $\ell_t = \min\{\phi_t, \upsilon(y^*)\}$.*

Proposition 5 establishes that the set of (dynamic) allocations in pure credit economies encompasses the set of allocations of pure monetary economies taking as given the trading mechanism. This result is related to those in Kocherlakota (1998), but with a key difference: while Kocherlakota (1998) shows that the set *all* implementable outcomes (allowing for arbitrary trading mechanisms) using money is contained in the set of all implementable outcomes with memory, we compare the equilibrium outcomes for the two economies under a *particular* trading mechanism. Later on we discuss the robustness to other trading mechanisms.

We illustrate this result in Figure 6 where the green, backward-bending line represents the first-order difference equation for a monetary equilibrium, (18), while the red area is the first-order difference inequality for a credit equilibrium, (11). Starting from some initial condition, d_0, we represent by a dashed line a sequence $\{d_t\}$ that satisfies the conditions for a monetary equilibrium. This sequence also satisfies the

[22] Due to the ex-ante heterogeneity between buyers and sellers it is strictly speaking the environment in Rocheteau and Wright (2005) or Nosal and Rocheteau (2011).

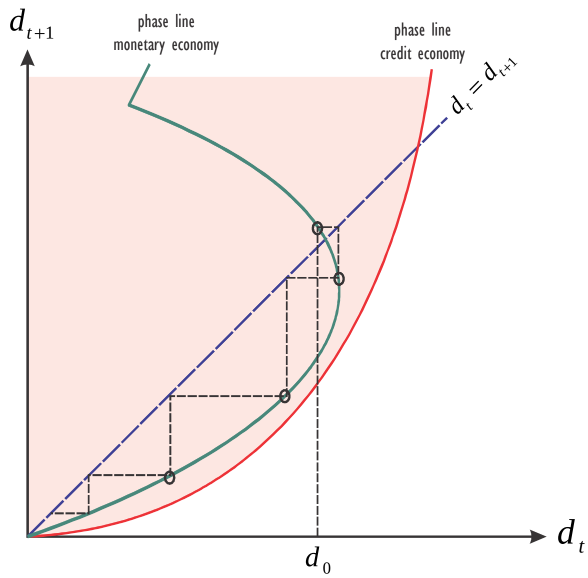

Figure 6: Monetary vs credit outcomes

conditions for a credit equilibrium, i.e., all pairs (d_t, d_{t+1}) are located in the red area. Therefore, if the equilibrium set of a pure monetary economy contains cycles and chaotic dynamics, the same must be true for the equilibrium set of the same economy with no money but record-keeping. It is also easy to see that credit equilibria in G2MW, under the "not-too-tight" solvency constraints requiring that (11) holds as an equality, do not coincide with the monetary equilibria in the Lagos-Wright economy since the phase line for credit economies differs from the phase line for monetary economies. The reason for this discrepancy is as follows. Under the AJ solvency constraints the payment capacity of buyers, d_t, is the discounted sum of all future match surpluses,

$$d_t = \beta \left\{ \alpha \left[u(y_{t+1}) - v(y_{t+1}) \right] + d_{t+1} \right\}.$$

In LW the payment capacity of buyers, ϕ_t, is the discounted sum of all future marginal surpluses multiplied by the value of money,

$$\phi_t = \beta \left\{ \alpha \frac{\partial \left[u \circ v^{-1}(\phi_{t+1}) - \phi_{t+1} \right]}{\partial \phi_{t+1}} \phi_{t+1} + \phi_{t+1} \right\}.$$

From the concavity of the match surplus, if $\phi_t = d_t$, then $\phi_{t+1} > d_{t+1}$.

The reverse of Proposition 5 does not hold; There are equilibria of pure credit economies that are not equilibria of pure monetary economies. As we saw above there are credit equilibria where trades shut down periodically, and such equilibria cannot be captured by Figure 6. (Recall that the recursive condition in Corollary 2 is sufficient but not necessary.) As another example, one can construct equilibria where the debt limit, d_t, increases in a monotonic fashion over time as buyers become more and more trustworthy. Such equilibria would not be sustainable in monetary economies.

3.4 Sunspot equilibria

So far we have focused on deterministic equilibria. One can also construct sunspot equilibria where the DM allocation, $\{(y_\chi, \ell_\chi)\}$, depends on the realization of a sunspot state, $\chi \in \mathbb{X}$, at the beginning of the DM. Suppose that $\mathbb{X}$ is finite and the process driving the sunspot state is iid with distribution π. We assume that π has a full support, i.e., $\pi(\chi) > 0$ for all $\chi \in \mathbb{X}$. The value of a buyer along the equilibrium path solves

$$V_\chi^b = \alpha\left[u(y_\chi) - \upsilon(y_\chi)\right] + \beta \bar{V}_\chi^b \tag{19}$$

$$\bar{V}_\chi^b = \int V_{\chi'}^b d\pi(\chi'), \tag{20}$$

for all $\chi \in \mathbb{X}$. As before, the lifetime utility of a buyer is the expected discounted sum of the surpluses coming from DM trades. It follows that a sunspot credit equilibrium is a vector, $\langle d_\chi; \chi \in \mathbb{X}\rangle$, that satisfies $d_\chi \leq \beta \bar{V}_\chi^b$. Hence, $\{d_\chi\}$ satisfies

$$rd_\chi \leq \alpha \int \{u[z(d_{\chi'})] - \upsilon[z(d_{\chi'})]\} d\pi(\chi') \quad \forall \chi \in \mathbb{X} \tag{21}$$

Proposition 6 ***(Sunspot equilibria)*** *Suppose that $\mathbb{X}$ has at least two elements and let π be a distribution over $\mathbb{X}$ with a full support. For a given $(\chi, d_\chi) \in \mathbb{X} \times (0, d^{\max})$, the set of sunspot credit equilibria with debt limit d_χ in state χ, denoted by $\Omega_{\mathbb{X},\pi}(\chi, d_\chi)$, has a positive Lebesgue measure in $\mathbb{R}^{|\mathbb{X}|-1}$.*

4 Alternative trading mechanisms

In the following we show that our results regarding the equilibrium set of pure credit economies are robust to trading mechanisms other than take-it-or-leave-it offers by buyers. We also extend our model in order to parametrize buyers' temptation to renege on their debt. This extension adds a new parameter that plays a key role for both the emergence of credit cycles under alternative trading mechanisms and the normative results in Section 5.

Suppose from now on that a buyer who promises to deliver ℓ units of goods in the next CM incurs the linear disutility of producing at the time he is matched in the DM. This new timing is illustrated in Figure 7.[23] The effort exerted by the buyer in the DM, ℓ, is perfectly observable to the seller. Hence, the seller can condition his own production to the buyer's effort. At the time of delivery, at the beginning of the CM, the disutility of production has been sunk and the buyer has the option to renege on his promise to deliver the good. The buyer's utility from consuming his own output is $\lambda\ell$ with $\lambda \leq 1$. A buyer has no incentive

[23] The description of the buyer's incentive problem is taken from Gu et al. (2013a,b).

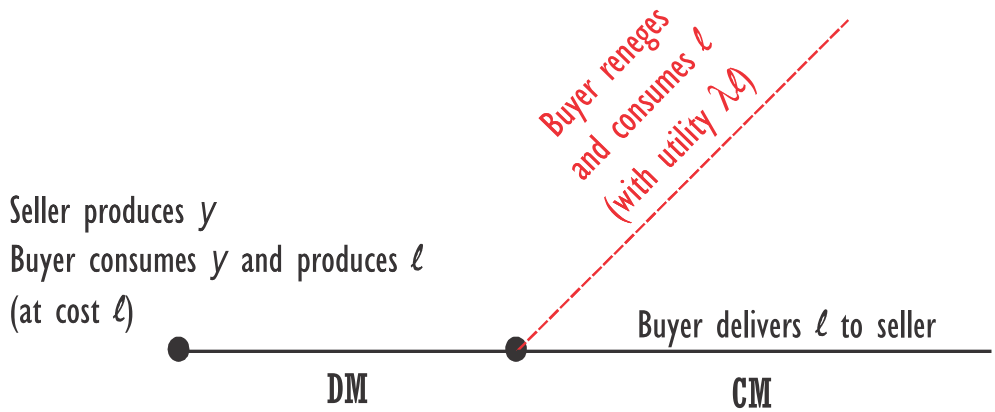

Figure 7: Timing of the extended model with temptation to renege

to produce more good than the amount he promises to repay to the seller since the net utility gain from producing x units of the good for oneself is $(\lambda - 1)x \leq 0$. Although the physical environment is different, mathematically speaking, the model of the previous section can be regarded as a special case with $\lambda = 1$. As before we will focus on symmetric Perfect Bayesian Equilibria with no default.[24]

Let $\{(d_t, y_t, \ell_t)\}_{t=0}^{\infty}$ be the sequence of equilibrium debt limits and trades. A necessary condition for the repayment of d_t to be incentive feasible is $\beta V_{t+1}^b \geq \lambda d_t$, where the left side is the buyer's continuation value from delivering the promised output and the right side of the inequality is the utility of the buyer if he keeps the output for himself, in which case he enjoys a utility flow λd_t, and goes to autarky. Following the same reasoning as before, a credit equilibrium is reduced to a sequence, $\{d_t\}_{t=0}^{\infty}$, that satisfies

$$\lambda d_t \leq \beta V_{t+1}^b = \alpha \sum_{s=1}^{+\infty} \beta^i \left[u(y_{t+s}) - \ell_{t+s} \right], \qquad t \in \mathbb{N}_0, \tag{22}$$

where the relationship between y_t, ℓ_t, and d_t will depend on the assumed trading mechanism.

4.1 Bargaining

It is standard in the literature on markets with pairwise meetings to determine the outcome of a meeting by an axiomatic bargaining solution. In this section we consider two well-known solutions: (i) the Kalai proportional bargaining solution and (ii) the generalized Nash solution. We adopt the representation of the equilibrium with solvency constraints, $\ell_t \leq d_t$, in order to obtain a convex bargaining set. For a given sequence of debt limits, $\{d_t\}_{t=0}^{+\infty}$, the buyer repays $\min\{\ell_t, d_t\}$ if his date-t obligation from his DM trade is ℓ_t.

[24] G2MW also introduce an imperfect record-keeping technology as follows. At the end of the CM of period t the repayments are recorded for a subset of buyers, $\mathbb{B}_t^r \subset \mathbb{B}$, chosen at random among all buyers. The set, $\mathbb{B}_t^r$, of monitored buyers is of measure π, and the draws from $\mathbb{B}$ are independent across periods. So in every period, while his promise is always recorded, a buyer has a probability π of having his repayment decision being recorded. Any equilibrium of our model with $\pi < 1$ is also an equilibrium with $\pi = 1$. Hence, setting $\pi = 1$ is with no loss in generality.

Due to the linearity of the CM value functions, the buyer's surplus from a DM trade, (y_t, ℓ_t) with $\ell_t \leq d_t$, is $u(y_t) - \ell_t$ and the seller's surplus is $-v(y_t) + \ell_t$.

Kalai proportional bargaining We amend the buyer take-it-or-leave-it offer game by restricting the set of buyers' feasible offers: the buyer can only make offers such that the fraction of the match surplus he receives is no greater than a given parameter $\theta \in [0, 1]$, i.e.,

$$u(y_t) - \ell_t \leq \theta[u(y_t) - v(y)]. \tag{23}$$

Thus, the buyer's offer in the DM, assuming he is in state G, solves

$$(y_t, \ell_t) = \arg\max_{y,\ell} [u(y) - \ell] \text{ s.t. (23) and } \ell \leq d. \tag{24}$$

According to (24) the buyer maximizes his utility of consumption net of the cost of repaying his debt subject to the feasibility constraint, (23), and the repayment constraint, $\ell \leq d$. The solution to (24) is

$$y_t = z(d_t) \equiv \min\{y^*, \eta^{-1}(d_t)\} \text{ and } \ell_t = \eta[z(d_t)]. \tag{25}$$

where $\eta(y) = (1 - \theta)u(y) + \theta v(y)$. In our proposed equilibrium, the buyer offers (y_t, ℓ_t) given by (25) and the seller accepts it. The seller rejects any offer from a buyer with state A.

Proposition 7 ***(Credit equilibrium under proportional bargaining)*** *A sequence, $\{d_t\}$, is a credit equilibrium under proportional bargaining if and only if*

$$\lambda d_t \leq \alpha\theta \sum_{i=1}^{+\infty} \beta^i \left[u(y_{t+i}) - v(y_{t+i})\right], \quad \forall t \in \mathbb{N}_0, \tag{26}$$

where (y_t, ℓ_t) is given by (25).

Proposition 7 describes the set of all debt limits, $\{d_t\}_{t=0}^{+\infty}$, and associated allocations, $\{(y_t, \ell_t)\}_{t=0}^{+\infty}$, that are generated by credit equilibria under bargaining weight θ. The right side of (26) takes into account that buyers only a receive a fraction θ of the match surplus. Note that Corollary 1 is a special case of Proposition 7 by taking $\theta = \lambda = 1$.

We can generalize Proposition 2 by showing that the set of steady-state equilibria is the interval $[0, d^{\max}]$, where $d^{\max}$ is the largest nonnegative root to $r\lambda d = \alpha\theta\{u[z(d)] - v[z(d)]\}$, and $d^{\max} > 0$ if and only if $\lambda r < \alpha\theta/(1 - \theta)$. If buyers do not have all the bargaining power, then an active steady-state credit equilibrium exists only if buyers are sufficiently patient.[25] The lower the value of θ the lower the rate of time

[25] The condition for existence of a credit equilibrium is the same as the condition for existence of a monetary equilibrium. See Nosal and Rocheteau (2011, Ch. 4.2.3).

preference that is required for credit to emerge. Indeed, if θ decreases buyers get a lower share in current and future match surpluses and, for a given d, the amount of DM consumption they can purchase is lower. Both effects reduce the gains from participating in the DM and hence reduce the maximum sustainable debt limit. It can also be checked that a higher λ reduces $d^{\max}$ since the temptation to renege on one's debt is higher. As a result any allocation under $\lambda = 1$ is also an allocation under $\lambda < 1$. We now move to equilibria with endogenous fluctuations.

Proposition 8 ***(2-Period Credit Cycles under proportional bargaining)*** *If $\lambda r < \alpha\theta/(1-\theta)$, then there exists a continuum of strict, 2-period, credit cycle equilibria. Moreover, if $r < \sqrt{1+\alpha\theta/[\lambda(1-\theta)]} - 1$, then there exist equilibria where credit shuts down periodically.*

Proposition 8 establishes a condition for the existence of a continuum of credit-cycle equilibria under proportional bargaining. The set of equilibria is represented by Figure 4 where the outer envelope shifts outward as θ increases. Moreover, if agents are sufficiently patient then there are equilibria where credit shuts down periodically, i.e., $\gamma(0) > 0$. In contrast, if we impose the AJ solvency constraints, then there are no periodic equilibrium under proportional bargaining, irrespective of the buyer's bargaining share. Propositions 4 and 5 regarding the existence of N-period credit cycles and the relationship between monetary and credit equilibria can be generalized to proportional bargaining in a similar fashion.

Generalized Nash bargaining Under generalized Nash bargaining the terms of the loan contract are

$$(y_t, \ell_t) \in \arg\max\, [u(y) - \ell]^{\theta} \, [\ell - \upsilon(y)]^{1-\theta} \quad \text{s.t.} \quad \ell \leq d_t.$$

The solution is given by (25) where

$$\eta(y) = \Theta(y)u(y) + [1-\Theta(y)]\,\upsilon(y) \text{ and } \Theta(y) = \theta\upsilon'(y)/\left[\theta\upsilon'(y) + (1-\theta)u'(y)\right]. \tag{27}$$

A sequence, $\{d_t\}_{t=0}^{+\infty}$, is a credit equilibrium under generalized Nash bargaining if and only if

$$\lambda d_t \leq \alpha \sum_{i=1}^{+\infty} \beta^i \left[u(y_{t+i}) - \eta(y_{t+i})\right], \quad \forall t \in \mathbb{N}_0, \tag{28}$$

where y_t is the solution to (25).

We denote $\hat{y} = \arg\max\{u(y) - \eta(y)\}$ the output level that maximizes the buyer's surplus. Unlike the proportional solution $\hat{y} < y^*$ for all $\theta < 1$. As a result the buyer's surplus, $u(y) - \eta(y)$, in the right side of the participation constraint, (28), is non-monotonic with the debt limit provided that $\theta < 1$.[26] It follows that

[26] This non-monotonicity property of the Nash bargaining solution and its implications for monetary equilibria is discussed at length in Aruoba, Rocheteau, and Waller (2007).

the function $\gamma(d)$ is non-monotonic, reaching a maximum at $d = \hat{d} \equiv \eta(\hat{y})$ and it is constant for $d > \eta(y^*)$. In Figure 8 we represent the function γ and the set of pairs, (d_0, d_1), consistent with a 2-period credit cycle equilibrium. One can see that the results are qualitatively unchanged except for the fact that the credit limits at a periodic equilibrium can be greater than the highest debt limit at a stationary equilibrium.[27] This result will have important normative implications.

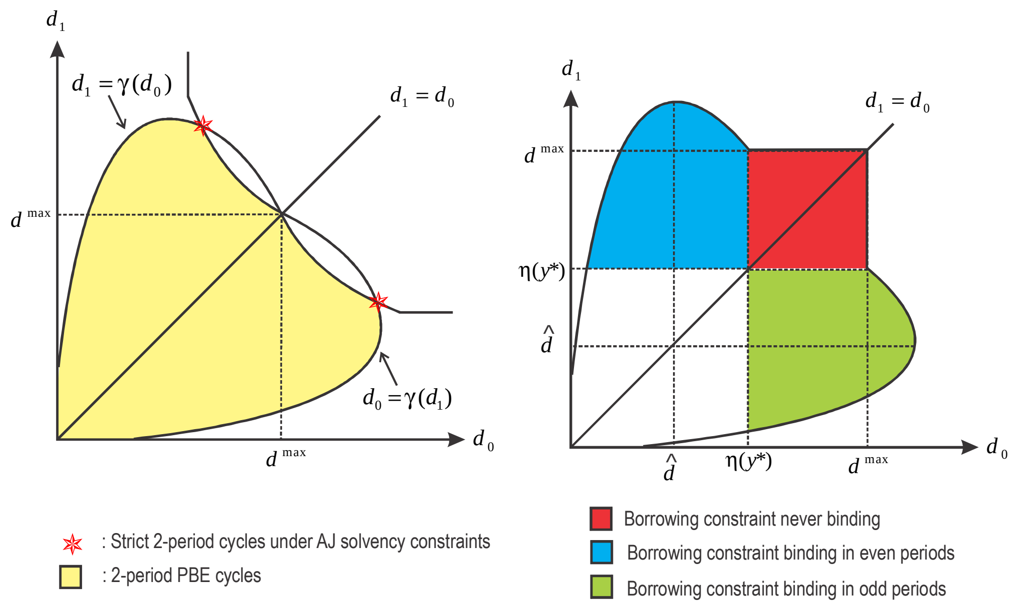

Figure 8: 2-period cycles under Nash bargaining

In the top panels of Figure 9 we plot the numerical examples in Gu and Wright (2011) under generalized Nash bargaining. The following functional forms and parameter values are $u(y) = [(x+b)^{1-a} - b^{1-a}]/(1-a)$ with $a = 2$ and $b = 0.082$, $v(y) = Ay$, $\beta = 0.6$, $\alpha = 1$, $\theta = 0.01$, and $\lambda = 3/40$. In the top left panel $A = 1.1$ and there are two 2-period cycles under "not-too-tight" solvency constraints, and they are such that borrowing constraints bind periodically. In the top right panel, $A = 1.5$ and the borrowing constraint binds in all periods. In contrast, for both examples there exists a continuum of PBE 2-period cycles, a fraction of which feature borrowing constraints that bind periodically and a fraction of which have borrowing constraints that binds in all periods.

4.2 Competitive pricing

Here we follow KL and AJ and assume that the terms of the loan contract in the DM are determined by price-taking behavior. We reinterpret matching shocks as preference and productivity shocks, i.e, only α

[27] In the Appendix we prove that any 2-period cycle under proportional bargaining is also a 2-period cycle under Nash bargaining.

buyers want to consume and only α sellers can produce. As in the previous sections, buyer's repayment strategy follows a threshold rule: for a given sequence of debt limits, $\{d_t\}_{t=0}^{+\infty}$, the buyer repays $\min\{\ell_t, d_t\}$ if his date-t obligation from his DM trade is ℓ_t. Moreover, the overall amount of debt issued by a buyer in the DM of period t, ℓ_t, is known to all agents. Hence, if p_t denotes the price of DM output in terms of the numéraire, buyer's demand is subject to the borrowing constraint, $p_t y \leq d_t$. For a given $\{d_t\}_{t=0}^{+\infty}$ the consistent market-clearing price is given by $p_t = v'(y_t)$, where

$$y_t = z(d_t) \equiv \min\{y^*, \eta^{-1}(d_t)\} \text{ and } \ell_t = \eta[z(d_t)], \tag{29}$$

with $\eta(y) = v'(y)y$.[28] The buyer's surplus is $u(y) - py = u(y) - v'(y)y$. For a given p, the buyer's surplus is non-decreasing in his borrowing capacity, d_t. However, once one takes into account the fact that $p = v'(y)$ then the buyer's surplus is non-monotone in his capacity to borrow, d_t. Provided that v is strictly convex, the buyer's surplus reaches a maximum for $y = \hat{y} < y^*$. A sequence, $\{d_t\}_{t=0}^{+\infty}$, is a credit equilibrium under competitive pricing if and only if (28) holds for all $t \in \mathbb{N}_0$, where y_t is given by (29). A steady state is a d such that

$$r\lambda d \leq \alpha \left\{u\left[z(d)\right] - v'\left[z(d)\right] z(d)\right\}. \tag{30}$$

Under some weak assumptions on v (for example, $\eta(y) = v'(y)y$ is convex), $d^{\max} > 0$, i.e., there exists a continuum of steady-state equilibria. This also implies that there exist a continuum of strict, 2-period, credit cycle equilibria.[29] This result can be contrasted with the ones in G2MW (Corollary 1-3) where conditions on parameter values are needed to generate a finite number (typically, two) of cycles. The right panel of Figure 8 illustrates these differences. Under "not-too-tight" solvency constraints credit cycles are determined at the intersection between $\gamma(d)$ and its mirror image with respect to the 45$^{\text{o}}$ line. These cycles are marked by a red star. If we allow for slack buyer's participation constraints, cycles are at the intersection of the area underneath $\gamma(d)$ and its mirror image with respect to the 45$^{\text{o}}$ line—the blue area in the figure. Finally, Proposition 5 on the equivalence result between monetary equilibria and credit equilibria holds for Walrasian pricing as well. (See the Appendix for a formal proof).

We now review the numerical examples in G2MW in the case where the DM market is assumed to be competitive. The functional forms are $u(y) = y$, $v(y) = y^{1+\gamma}/(1+\gamma)$, and there are no idiosyncratic shocks, $\alpha = 1$. The first example in the bottom left panel of Figure 9 is obtained with the following parameter values:

[28] The buyer's problem is $\max_y \{u(y) - p_t y\}$ s.t. $p_t y \leq d_t$. The solution is $y_t = \min\left\{u'^{-1}(p_t), d_t/p_t\right\}$. Using that there is the same measure, α, of buyers and sellers participating in the market, market clearing implies $p_t = v'(y_t)$. As a result $y_t = y^*$ if $y^* v'(y^*) \geq d_t$ and $y_t v'(y_t) = d_t$ otherwise. For a detailed description of this problem in the context of a pure monetary economy, see Rocheteau and Wright (2005, Section 4).

[29] Under competitive pricing, the function γ (analogous to (14)) may not be monotone or concave, but the logic for Proposition 3 does not depend on those properties. See also the supplemtary appendix S2 for a formal proof of the existence of 2-period cycles.

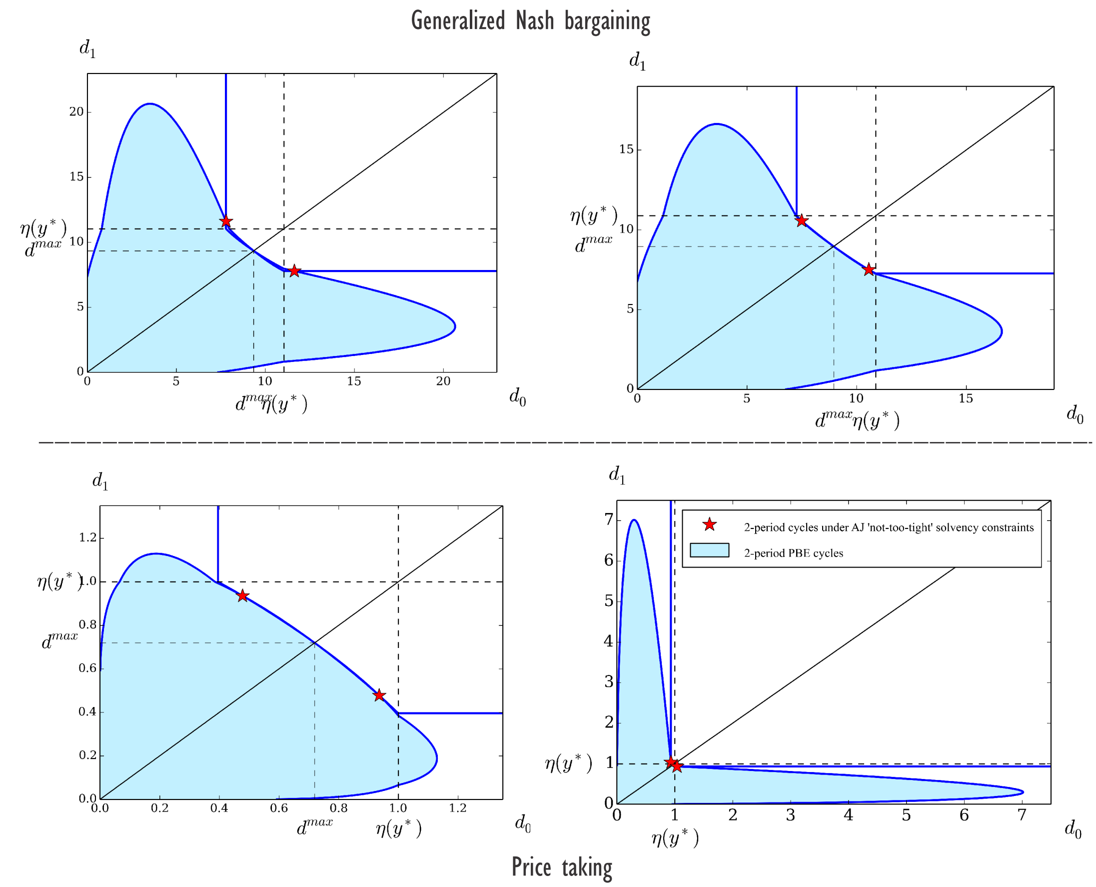

Figure 9: The blue area is the set of all PBE credit cycles. The red dots are credit cycles under AJ "not-too-tight" solvency constraints. The top panels are obtained under generalized Nash bargaining while the bottom panels are obtained under price taking.

$\gamma = 2.1$, $\beta = 0.4$, $\lambda = 1/6$. Under the AJ solvency constraints G2MW identify two (strict) two-period cycles, $(d_0, d_1) = (0.477, 0.936)$ and its converse, marked by red dots in the figure. The second example in the bottom right panel is obtained with the following parameter values: $\gamma = 0.5$, $\beta = 0.9$, $\lambda = 1/10$. The credit cycles under the AJ solvency constraints, $(d_0, d_1) = (0.933, 1.037)$ and its converse, are such that period allocations fluctuate between being debt-constrained and unconstrained. We find a much bigger set of PBE credit cycles represented by the blue colored region composed of a continuum of cycles in which the allocations fluctuate between being debt-constrained and unconstrained and a continuum of cycles in which agents are debt-constrained in all periods. In the second example, the credit cycle under AJ solvency constraints is such that $(y_0, y_1) = (0.96, 1.00)$ while the most volatile PBE is $(y_0, y_1) = (0.96, 0.00)$.

5 Normative analysis

We now turn to the normative implications of our model. We will characterize constrained-efficient allocations under two alternative market structures: pairwise meetings and large-group meetings. We will show that under pairwise meetings the optimal mechanism is the one studied in Section 3 and "not-too-tight" solvency constraints are socially optimal. Under large-group meetings the optimality of "not-too-tight" solvency constraints depends on buyers' temptation to renege (λ). When λ is large the AJ Second Welfare Theorem holds, i.e., constrained-efficient allocations are implemented with "not-too-tight" solvency constraints. In contrast, when λ is small, constrained-efficient allocations are non-stationary and feature slack participation constraints. We will give examples where credit cycles with "too-tight" solvency constraints dominate cycles and steady states under "not-too-tight" constraints. Finally, we will end this section by characterizing the welfare-maximizing PBE under given trading mechanisms (e.g., Nash bargaining).

5.1 Optimal mechanism with pairwise meetings

We study the problem of a planner who chooses the allocations, $\{(y_t, \ell_t)\}_{t=0}^{+\infty}$, in order to maximize the discounted sum of all match surpluses subject to incentive-feasibility conditions:[30]

$$\max_{\{(y_t,\ell_t)\}} \sum_{t=0}^{+\infty} \beta^t \alpha \left[u(y_t) - \upsilon(y_t)\right] \tag{31}$$

$$\text{s.t.} \quad \lambda \ell_t \leq \sum_{s=1}^{+\infty} \beta^s \alpha \left[u(y_{t+s}) - \ell_{t+s}\right] \tag{32}$$

$$\upsilon(y_t) \leq \ell_t \leq u(y_t). \tag{33}$$

The inequality, (32), is the participation constraint guaranteeing that buyers prefer to repay their debt rather than going to permanent autarky. The conditions in (33) make sure that both buyers and sellers receive a positive surplus from their DM trades. Coalition-proofness in pairwise meetings requires that $y_t \leq y^*$, which is satisfied endogenously (and hence ignored thereafter). We call a solution to (31)-(33) a constrained-efficient allocation (c.e.a.). In the following we define $y^{\max}$ as the highest, stationary level of output consistent with both the seller's and buyer's participation constraints. It is the positive solution to $\lambda r \upsilon(y^{\max}) = \alpha \left[u(y^{\max}) - \upsilon(y^{\max})\right]$.

Proposition 9 ***(c.e.a. under pairwise meetings)***

[30] Kocherlakota (1996) and Gu et al. (2013a, Section 7) study a Pareto problem to determine a contract curve linking the expected discounted utilities of buyers and sellers. In contrast the planner's objective in our model is a social welfare function that aggregates the buyers' and sellers' utilities. One can interpret this social welfare function as the ex ante expected utility of a representative agent in a version of the model where the role of an agent in the DM is determined at random in each period.

1. If $y^* \leq y^{\max}$, then any c.e.a. is such that $y_t = y^*$ and $\ell_t \in [\upsilon(y^*), \bar{\ell}]$ for all $t \in \mathbb{N}_0$, where $\bar{\ell} = \alpha\left[u(y^*) - \upsilon(y^*)\right]/\lambda r$.

2. If $y^* > y^{\max}$, then c.e.a. is such that $y_t = y^{\max}$ and $\ell_t = \upsilon(y_t)$ for all $t \in \mathbb{N}_0$.

If $\lambda r \leq \alpha\left[u(y^*)/\upsilon(y^*) - 1\right]$, then the first-best allocation is implementable.[31] This condition holds if agents are sufficiently patient (r low) and if the temptation to renege is not too large (λ low). In contrast, if $\lambda r > \alpha\left[u(y^*)/\upsilon(y^*) - 1\right]$, then the constraint-efficient outcome is $y_t = y^{\max} < y^*$, which corresponds to the highest steady state.

The c.e.a. in Proposition 9 can be implemented by having buyers set the terms of the loan contract unilaterally, in which case $\ell_t = \upsilon(y_t)$ for all t.[32] By giving all the bargaining power to buyers the planner relaxes participation constraints in the CM, which allows for more borrowing and higher levels of output. Moreover, the solvency constraint in the buyer's bargaining problem must be "not-too-tight," in accordance with AJ's welfare theorem. We summarize this implementation result in the following Corollary.

Corollary 5 ***(Second Welfare Theorem for economies with pairwise meetings)*** *The c.e.a. is implemented with take-it-or-leave-it offers by buyers under "not-too-tight" solvency constraints.*

5.2 Optimal mechanism with centralized meetings

Suppose next that agents meet in a centralized location in the DM. If we do not allow cooperative defections, then Proposition 9 holds and $\ell_t = \upsilon(y_t)$. However, if $\upsilon'' > 0$, this restriction is binding. Indeed, a buyer and two sellers can form a deviating coalition in which each seller produces $y_t/2$ at a total cost of $2\upsilon(y_t/2) < \upsilon(y_t)$ and the buyer compensates the sellers by offering them a positive surplus, $\ell_t/2 - \upsilon(y_t/2) > 0$. In order to prevent such defections we impose the core requirement in the DM or, equivalently, the competitive equilibrium outcome.[33] Hence, from (29) the terms of the loan contract are given by $\ell = \eta(y) = \upsilon'(y)y$.

The planner's problem is similar to (31)-(33) where the buyer's participation constraint in the CM, (32), is replaced with

$$\lambda\eta(y_t) \leq \alpha \sum_{s=1}^{+\infty} \beta^s \left[u(y_{t+s}) - \eta(y_{t+s})\right], \tag{34}$$

[31] Hu, Kennan, and Wallace (2009) derive the same condition for pure monetary economies in the case where $\lambda = 1$. A difference, however, is that the game where buyers make take-it-or-leave-it offers is not the optimal mechanism in monetary economies.

[32] Notice, however, that the c.e.a. is not uniquely implemented by the optimal mechanism. Indeed, an equilibrium under take-it-or-leave-it offers by buyers and "not-too-tight" solvency constraints solves $d_t = \beta\left\{\alpha\left[u(y_{t+1}) - \upsilon(y_{t+1})\right]/\lambda + d_{t+1}\right\}$. There are a continuum of equilibria converging to the autarky steady state. The c.e.a. is the only bounded sequence that does not converge to the autarky equilibrium. For related results in the context of the AJ model see Bloise, Reichlin, and Tirelli (2013).

[33] See Hu, Kennan, and Wallace (2009) and Wallace (2013) for a related assumption in the context of monetary economies. The equivalence result between the core and competitive equilibrium allocations for economies with a continuum of agents was first shown by Aumann (1964). See supplementary appendix $S3$ for this equivalence result in the context of our model.

and the DM participation constraints, (33), are satisfied by definition of η for all $y \in [0, y^*]$. We define two critical values for DM output:

$$\hat{y} = \arg\max_{y \in [0,y^*]} [u(y) - \eta(y)] \tag{35}$$

$$y^{\max} = \max\{y > 0 : \alpha[u(y) - \eta(y)] \geq r\lambda\eta(y)\}. \tag{36}$$

The quantity $\hat{y}$ is the output level that maximizes the buyer's surplus in the DM. The quantity $y^{\max}$ is the highest, stationary level of output that is consistent with the buyer's participation constraint in the CM. We assume that both $\hat{y}$ and $y^{\max}$ are well-defined and, for all $0 \leq y \leq y^{\max}$, $\alpha[u(y) - \eta(y)] \geq r\lambda\eta(y)$.

Proposition 10 ***(c.e.a. under centralized meetings: Part I)***

1. *If $y^* \leq y^{\max}$, then the c.e.a. is such that $y_t = y^*$ and $\ell_t = \eta(y^*)$ for all $t \in \mathbb{N}_0$.*

2. *If $y^{\max} \leq \hat{y} \leq y^*$, then the c.e.a. is such that $y_t = y^{\max}$ and $\ell_t = \eta(y^{\max})$ for all $t \in \mathbb{N}_0$.*

In accordance with "Folk theorems" for repeated games, provided that agents are sufficiently patient, $r \leq \alpha[u(y^*) - \eta(y^*)]/\lambda\eta(y^*)$, the first-best allocation is an equilibrium outcome. If $y^{\max} \leq \hat{y}$ (second part of Proposition 10) the first best is not implementable. As shown in the left panel of Figure 10, the buyer's welfare and society's welfare are both increasing with y over $(0, y^{\max})$. Hence, the highest steady state maximizes social welfare.

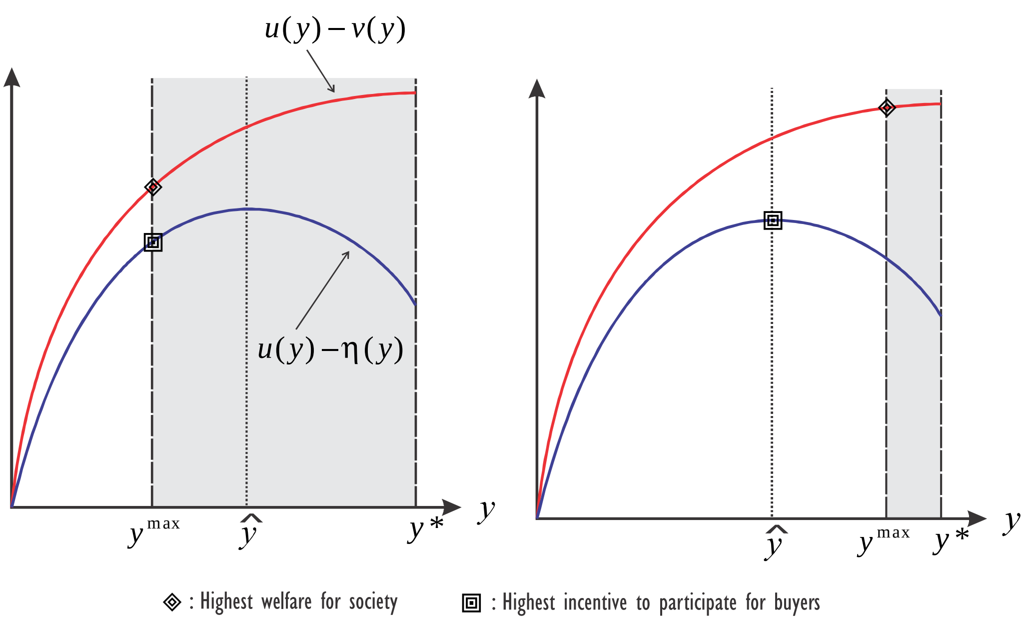

Figure 10: Let panel: No trade-off between efficiency and incentives over $[0, y^{\max}]$; Right panel: A trade-off between efficiency and incentives over $[\hat{y}, y^*]$.

We now turn to the case where $\hat{y} < y^{\max} < y^*$. For all $y \in (\hat{y}, y^{\max})$ the buyer's surplus, $u(y) - \eta(y)$, and society's surplus, $u(y) - v(y)$, covary negatively with y, as shown in the right panel of Figure 10. This negative relationship gives rise to a trade-off between social efficiency and incentives for debt repayment. As a result of this trade-off the highest steady state, $y^{\max}$, might no longer be the PBE outcome that maximizes social welfare.

In order to analyze this trade-off formally we write the problem recursively by introducing the buyer's "promised utility," ω_t, as a new state variable.[34] Society's welfare, denoted $V(\omega)$, solves the following Bellman equation,

$$V(\omega) = \max_{y,\omega'} \{\alpha [u(y) - v(y)] + \beta V(\omega')\} \tag{37}$$

$$\text{s.t.} \quad -\eta(y) + \beta \frac{\omega'}{\lambda} \geq 0 \tag{38}$$

$$\omega' \geq (1+r) \{\omega - \alpha [u(y) - \eta(y)]\} \tag{39}$$

$$y \in [0, y^*], \ \omega' \in [0, \bar{\omega}], \tag{40}$$

where $\bar{\omega} = [u(\hat{y}) - \eta(\hat{y})] / (1 - \beta)$ is the maximum lifetime expected utility a buyer can expect across all PBE. The novelty is the promise-keeping constraint, (39), according to which the lifetime expected utility promised to the buyer along the equilibrium path, ω, is implemented by generating an expected surplus in the current period equal to $\alpha [u(y) - \eta(y)]$ and by making a promise for future periods worth $\beta\omega'$. In the appendix, we show that there is a unique V solution to (37)-(40) in the space of continuous, bounded and concave functions, and this solution is non-increasing. As a result, the maximum value for society's welfare is $V(0) = \max_{\omega \in [0,\bar{\omega}]} V(\omega)$, as the initial promised utility to the buyer is a choice variable. The next Proposition provides a full analytical characterization of this problem.

Proposition 11 ***(c.e.a. under centralized meetings: Part II)*** *Suppose that $\hat{y} < y^{\max} < y^*$ and that η is a convex function.*

1. *If $\lambda \geq \alpha [1 - u'(y^{\max})/\eta'(y^{\max})]$, then the c.e.a. is such that $y_t = y^{\max}$ and $\ell_t = \eta(y^{\max})$ for all $t \in \mathbb{N}_0$.*

2. *If $\lambda < \alpha [1 - u'(y^{\max})/\eta'(y^{\max})]$, then the c.e.a. is such that $y_0 \in (y^{\max}, y^*)$ and $y_t = y_1 \in (\hat{y}, y^{\max})$ for all $t \geq 1$, where (y_0, y_1) is the unique solution to*

$$\max_{y_0,y_1} \left\{ u(y_0) - v(y_0) + \frac{u(y_1) - v(y_1)}{r} \right\} \tag{41}$$

$$s.t. \quad \eta(y_0) = \frac{\alpha[u(y_1) - \eta(y_1)]}{\lambda r}. \tag{42}$$

[34] Our recursive formulation is very close to the self-generation technique in Abreu, Pearce, and Stacchetti (1990), which characterizes the set of payoffs generated by Perfect Public Equilibria. In contrast to their analysis, we are able to pin down the set of allocations as well, and, in particular, the debt limits.

Equation (42) is the buyer's participation constraint at $t = 0$ if output is constant and equal to y_1 in all future periods, which is shown to be socially optimal. Provided that $y_1 > \hat{y}$, this participation constraint gives a trade-off between current and future output. In the neighborhood of the highest steady state:

$$\left.\frac{dy_1}{dy_0}\right|_{y^{\max}} = \frac{\lambda r \eta'(y^{\max})}{\alpha[u'(y^{\max}) - \eta'(y^{\max})]} < 0.$$

When $\lambda \geq \alpha\,[1 - u'(y^{\max})/\eta'(y^{\max})]$ exploiting this trade-off is harmful to society's welfare since one would have to implement a large drop in future output below $y^{\max}$ in order to raise current output by a small amount above $y^{\max}$ while maintaining the buyer's incentive to repay his debt.

In contrast, when λ is small, it is optimal to exploit the trade-off between current and future output arising from (42). The optimal allocation is such that initial output, y_0, is larger than $y^{\max}$ while future output, y_1, is constant and lower than $y^{\max}$. Because $y_1 < y^{\max}$ society's welfare in future periods, $u(y_1) - v(y_1)$, is lower than its level at the highest steady state, $u(y^{\max}) - v(y^{\max})$. However, the buyer's surplus is higher, $u(y_1) - \eta(y_1) > u(y^{\max}) - \eta(y^{\max})$, which relaxes the incentive constraint for repayment in the initial period, and hence raises current output, $y_0 > y^{\max}$, and current society's welfare, $u(y_0) - v(y_0) > u(y^{\max}) - v(y^{\max})$.[35]

In Figure 11 we illustrate the determination of (y_0, y_1). The red curve labelled IR corresponds to (42), the debt limit at $t = 0$ given the output level in subsequent periods. As y_1 increases above $\hat{y}$ the debt limit decreases and hence y_0 decreases as well. By definition the IR curve intersects the 45$^{\text{o}}$-line at $y^{\max}$. The blue curve labelled EULER corresponds to the first-order condition of the problem (41)-(42). Given the strict concavity of the surplus function it is optimal to smooth consumption by increasing y_0 when y_1 increases. When λ is low the EULER curve is located above the IR curve at $y_1 = y^{\max}$. As a result, the optimal solution, denoted (y_0^{**}, y_1^{**}) is such that $y_0^{**} > y^{\max}$ and $y_1^{**} < y^{\max}$.

Corollary 6 ***(Second Welfare Theorem under large-group meetings)*** *Assume that η is a convex function.*

1. *If either $y^{\max} \leq \hat{y} \leq y^*$ or $\hat{y} < y^{\max} < y^*$ and $\lambda \geq \alpha\,[1 - u'(y^{\max})/\eta'(y^{\max})]$, then the c.e.a. is implemented with "not-too-tight" solvency constraints.*

2. *If $\hat{y} < y^{\max} < y^*$ and $\lambda < \alpha\,[1 - u'(y^{\max})/\eta'(y^{\max})]$, then the c.e.a. is implemented with slack repayment constraints in all future periods, $t \geq 1$.*

The first part of Corollary 6 establishes conditions under which the AJ Second Welfare Theorem applies to our credit economy, namely, the c.e.a. is implemented with "not-too-tight" solvency constraints. A sufficient

[35] KL provide an example where partial exclusion leads to a welfare-improving outcome. See their Example 2 on p. 875.

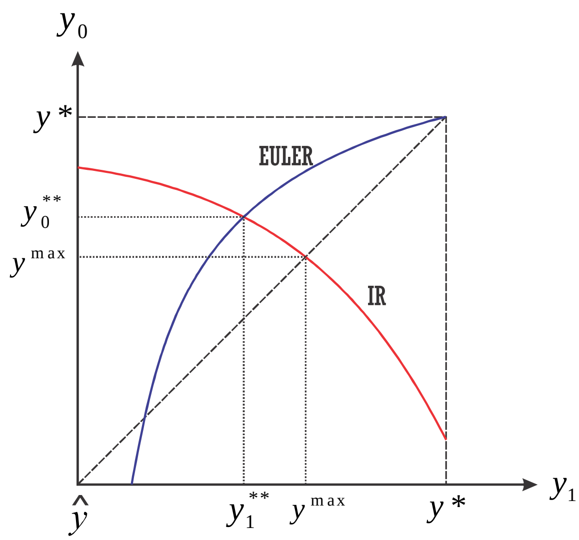

Figure 11: Determination of the constrained-efficient allocation, (y_0, y_1)

condition is $y^{\max} \leq \hat{y}$ so that there is no trade-off between efficiency and incentive. If $\hat{y} < y^{\max} < y^*$, it is socially optimal to keep solvency constraints "not-too-tight" provided that λ is sufficiently large.

If $\hat{y} < y^{\max} < y^*$ and λ is small, then the c.e.a. is implemented with participation constraints that are slack in all future periods (second part of Corollary 6). The failure of the AJ Welfare Theorem is surprising as one would conjecture that higher debt limits allow society to generate larger gains from trade. This reasoning is valid in a static sense. If d_t increases, the sum of all surpluses in period t, $\alpha\left[u(y_t) - v(y_t)\right]$, increases. However, there is a general equilibrium effect associated with a higher d_t, i.e., more IOUs are competing for DM goods. This increased competition raises the price of DM goods, $p_t = v'(y_t)$. If the economy is close enough to the first best, this pecuniary externality lowers the buyers' welfare (even though society as a whole is better off) and worsens their incentive to repay their debt in earlier periods.

So far we allowed the planner to pick the equilibrium among the full set of PBE. The results in Proposition 11 are robust if we restrict the planner to choose among 2-period cycles. To see this we adopt the numerical example from the left panel of Figure 9, $\gamma = 2.1$, $\beta = 0.4$, $\lambda = 1/6$. For these parameter values $\lambda < \alpha\left[1 - u'(y^{\max})/\eta'(y^{\max})\right]$. Society's welfare over a 2-period cycle is measured by $u\left[y(d_0)\right] - v\left[y(d_0)\right] + \beta\left\{u\left[y(d_1)\right] - v\left[y(d_1)\right]\right\}$. In the left panel of Figure 12 we highlight in red the set of 2-period cycles, (d_0, d_1), that dominate the equilibria under "not-too-tight" solvency constraints. There exist a continuum of such cycles that feature slack participation constraints. Moreover, we represent by a green area the set of 2-period cycles that dominate the highest steady state, $y^{\max}$. We also find a continuum of such cycles. Hence, the imposition of "not-too-tight" solvency constraints eliminates good equilibria. Finally, we represent with a

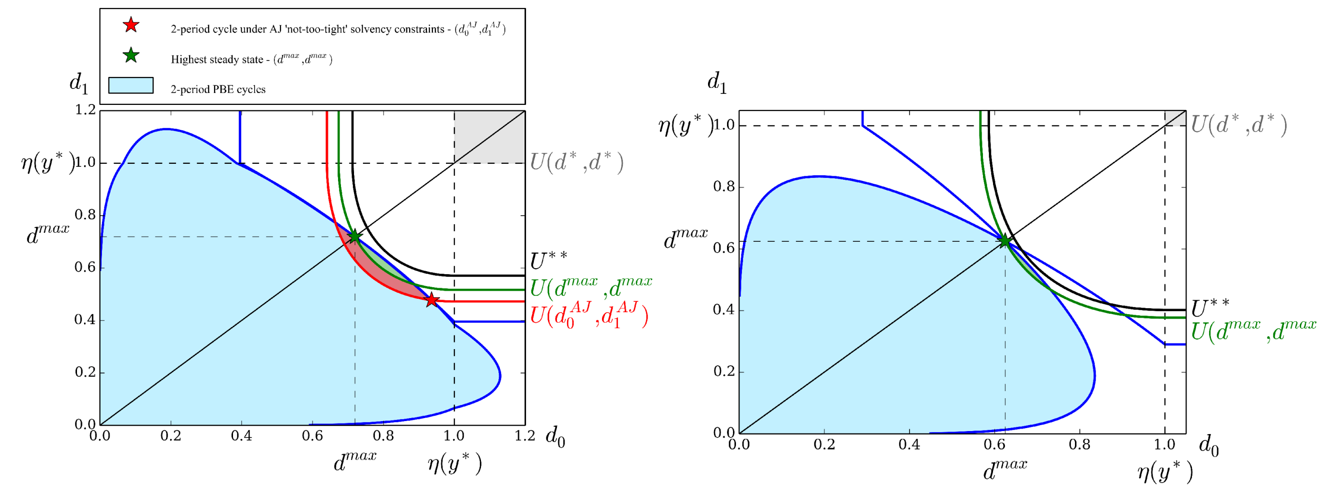

Figure 12: The blue area is set of all 2-period cycles. The red star is the 2-period cycle in G2MW and the green star is the highest steady state. Left panel: $\lambda = 1/6$; Right panel: $\lambda = 1/4$.

black indifference curve society's welfare at the constrained-efficient allocation and by a grey area the set of debt limits that implement the first best. Figure 12 shows that the first best is not implementable and, in accordance with the second part of Proposition 11, the constrained-efficient allocation is not a 2-period cycle.

The right panel of Figure 12 considers the same example as above when λ is reduced from $\lambda = 1/6$ to $\lambda = 1/4$. The condition in Part 2 of Proposition 11 holds so that the highest steady state is not constrained-efficient. Moreover, there is no credit cycle under the "not-too-tight" solvency constraints. However, there are a continuum of cycles under "too-tight" constraints, a fraction of which dominate the highest steady state.

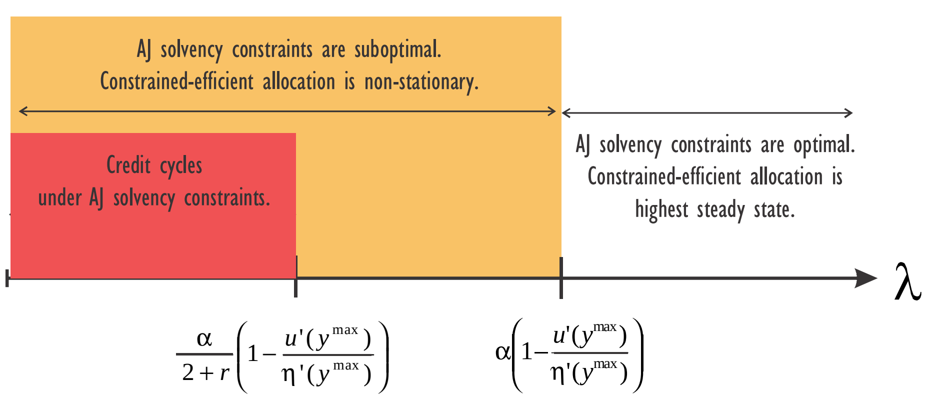

Figure 13: Optimality of AJ solvency constraints and cycles

Finally, G2MW established a sufficient condition, $\lambda < \alpha \left[1 - u'(y^{\max})/\eta'(y^{\max})\right]/(2+r)$, for the emergence of endogenous credit cycles under the "not-too-tight" solvency constraint. As illustrated in Figure 13 this condition is incompatible with the condition in Proposition 11 for "not-too-tight" solvency constraints to implement a c.e.a., i.e., $\lambda \geq \alpha \left[1 - u'(y^{\max})/\eta'(y^{\max})\right]$.

5.3 Optima under arbitrary trading mechanism

We now turn to the optimal credit equilibrium allocation taking the mechanism to set the terms of the loan contract, η, as given. Although Propositions 10-11 are obtained under competitive pricing, they hold for any arbitrary trading mechanism, η. For example, if η is determined by proportional bargaining, then $\hat{y} = y^*$, and Proposition 10 implies that the best PBE corresponds to the highest steady state, $y_t = y^{\max}$ for all t. Proposition 10 also applies to generalized Nash bargaining: if $y^{\max} \leq \hat{y} \leq y^*$ or $y^* \leq y^{\max}$, then the best PBE is the highest steady state (in the proof of the proposition we only use the fact that $\hat{y}$ is the unique maximizer). However, under Nash bargaining, the loan contract η may not be convex in general, and hence Proposition 11 may not apply. Nevertheless, we showed that (37)-(40) defines a contraction mapping so that we can easily solve for the best PBE allocation numerically.

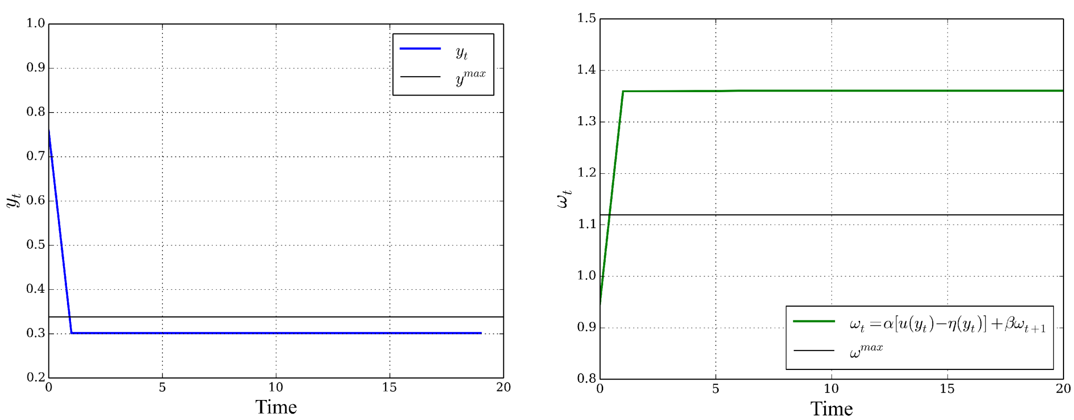

Figure 14: Best PBE under Nash bargaining

In Figure 14 we adopt the same functional forms and parameter values as the ones in the bottom left panel of Figure 9. The left panel plots the DM output level over time while the right panel plots the buyer's lifetime expected discounted utility. It can be seen that the allocation that maximizes social welfare is non-stationary. The output level is high in the first period and low and constant in all subsequent periods, in accordance with the second part of Proposition 11. Conversely, the buyer's lifetime expected utility is low initially and high afterwards. The logic for why the solution is non-stationary is similar to the one described in the case of price taking. Given that the buyer's surplus is hump-shaped, one can implement a high level of output in the initial period by promising a high utility in the future, which is achieved by lowering future output. In Figure 15 we represent the set of 2-period cycles under this parametrization. There are a continuum of cycles that dominate the periodic equilibria obtained under AJ solvency constraints (the red area) and the highest steady state (the green area).

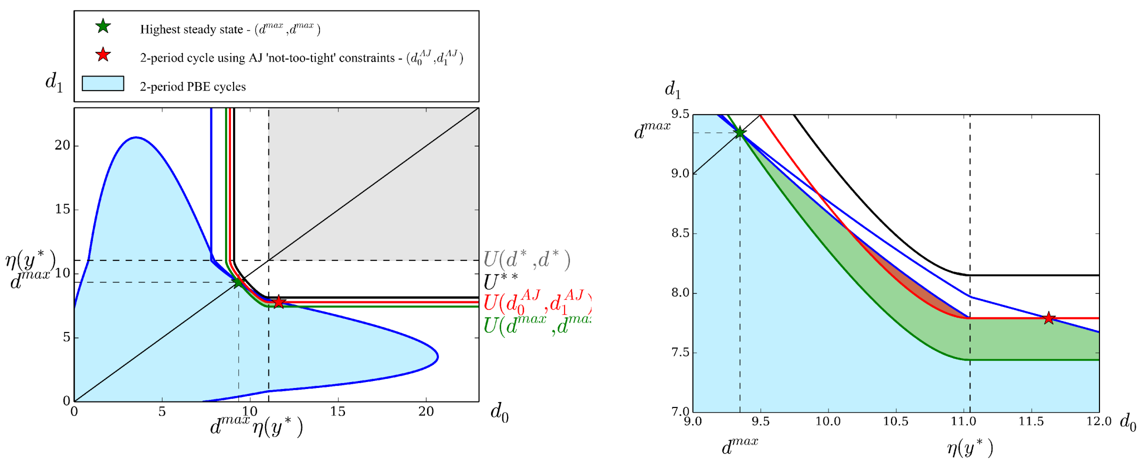

Figure 15: 2-period cycles and their welfare properties under Nash bargaining

6 Conclusion

We have characterized the set of outcomes of a pure credit economy and their welfare properties. The economy features intertemporal gains from trade that can be exploited with simple intra-period loan contracts. Such contracts and their execution are publicly recorded. Agents interact either through random, pairwise meetings under various trading mechanisms, as in the New-Monetarist literature, or in competitive spot markets, as in AJ. In sharp contrast with the existing literature (e.g., G2MW) we have shown that such economies exhibit a continuum of steady states, a continuum of endogenous cycle equilibria of any periodicity, and a wide variety of non-stationary equilibria. We have shown that any outcome of a pure monetary economy with no record-keeping but fiat money—which includes cycles and chaotic dynamics—is also an outcome of the pure credit economy. The reverse is not true. There are outcomes of pure credit economies that cannot be implemented by monetary equilibria. For instance, there are equilibria where credit shuts down periodically while there are no equilibria where fiat money is valueless periodically. We also provided examples where steady states are dominated in terms of social welfare by cycles where borrowing constraints are slack periodically. Finally, we have characterized the PBEs that maximize social welfare and we showed that when the temptation to renege is small constrained-efficient allocations are implemented with slack participation constraints for borrowers.

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APPENDIX: Proofs of lemmas and propositions

Proof of Proposition 1 **(⇒)** Here we prove necessity. Suppose that $\{(y_t, x_t, \ell_t)\}_{t=0}^{\infty}$ is an equilibrium outcome in a credit equilibrium, (s^b, s^s).

(i) Here we show condition (4). Because the worst payoff to buyers at each period is 0 (autarky) while the equilibrium payoff at period t is $u(y_t) - x_t$, condition (4) is necessary for buyers to repay their promises at each period.

(ii) To show condition (5), we first show that $x_t = v(y_t)$ for all t. Note that (A3) implies that $x_t = \ell_t$ for all t. If $x_t < v(y_t)$, then the seller would not accept the offer. Suppose, by contradiction, that $x_t > v(y_t)$. Then, the buyer may deviate and offer (y', ℓ_t) with $v(y') \in (v(y_t), \ell_t)$. Because this deviation does not affect the buyer's public record and the buyer has the same incentive to repay his debt, it is dominant for the seller to accept it. It then is a profitable deviation because $y' > y_t$.

Next, to show that $y_t \le y^*$ for all t, suppose, by contradiction, that $y_t > y^*$ and hence $u(y_t) \ge x_t \ge v(y_t) > v(y^*)$. Then there exists an alternative offer, $(y', \ell') = (y', x')$, such that $u(y') - x' > u(y_t) - x_t$ and $-v(y') + x' > -v(y_t) + x_t$ and $\ell' \le \ell_t$. It is dominant for the seller to accept this alternative offer. The seller's payoff at the current period is 0 if he rejects. However, if he accepts, then by (A3), the threshold rule for repayment, the buyer will repay his promise $\ell' = x'$. Then, by accepting the offer the seller obtains $-v(y') + x' > 0$. Thus, (y', ℓ') is a profitable deviation for the buyer.

(⇐) Here we show sufficiency. Let $\{(y_t, x_t, \ell_t)\}_{t=0}^{\infty}$ be a sequence satisfying (4) and (5). Consider (s^b, s^s) given as follows. Buyers can be in two states, $\chi_{i,t} \in \{G, A\}$, and each buyer's initial state is $\chi_{i,0} = G$. The law of motion of the buyer i's state are given by:

$$\chi_{i,t+1}\left[(\ell', x', i), \chi_{i,t}\right] = \begin{cases} A & \text{if } x' < \min(x_t, \ell') \text{ or } \chi_{i,t} = A \\ G & \text{otherwise} \end{cases} . \tag{43}$$

The strategies are such that $s^b_{t,1}(\rho^{i,t}) = (y_t, \ell_t)$ if the state for $\rho^{i,t}$ is G and $s^b_{t,1}(\rho^{i,t}) = (0,0)$ otherwise; $s^b_{t,2}(\rho^{i,t}, (y', \ell'), yes) = \min\{\ell', \ell_t\}$ if the state for $\rho^{i,t}$ is G and $s^b_{t,2}(\rho^{i,t}, (y', \ell'), yes) = 0$ otherwise; $s^s_t(\rho^{i,t}, (y', \ell')) = yes$ if the state for $\rho^{i,t}$ is G and $v(y') \le \min\{\ell', \ell_t\}$, and $s^s_t(\rho^{i,t}, (y', \ell')) = no$ otherwise. We show that (s^b, s^s) is a credit equilibrium.

Given s^b, s^s is optimal: the seller expects a buyer in state G to repay up to ℓ_t at period t and hence he accepts an offer, (y', ℓ'), if $v(y') \le \min\{\ell', \ell_t\}$; with buyers in state A he expects no repayment at all and hence rejects any offer. Next, we show that s^b is optimal given s^s. Consider a buyer with state A at the beginning of period t. Any offer to the seller is rejected and therefore it is optimal for the buyer to offer $(0, 0)$. Similarly, for such a buyer at the CM stage at period t with a promise ℓ', his state will remain in A, independent of his repayment decision and hence it is optimal to repay nothing.

Now consider a buyer with state G at the CM stage of period t, with a promise ℓ' made to the seller. The buyer has to pay $\min\{\ell_t, \ell'\}$ to maintain state G. By (4), paying this amount is better than becoming a A person, whose continuation value is 0. Finally, consider a buyer with state G at the beginning of period t. Note that under s^b, his continuation value from period $t+1$ onward is independent of his offer at period t. Moreover, for any offer (y, ℓ), the seller accepts the offer if and only if $v(y) \leq \min\{\ell, \ell_t\}$. Thus, the buyer problem is

$$\max_{(y,\ell)} u(y) - \min\{\ell, \ell_t\} \text{ s.t. } v(y) \leq \min\{\ell, \ell_t\}.$$

Because $\ell_t = v(y_t) \leq v(y^*)$, (y_t, ℓ_t) is a solution to the problem. $\square$

Proof of Corollary 1 **($\Leftarrow$)** Here we show sufficiency. Let $\{d_t\}_{t=0}^{\infty}$ be a sequence satisfying (9) and (10). Then, we can determine the outcome, $\{(y_t, x_t, \ell_t)\}_{t=0}^{\infty}$, consistent with $\{d_t\}_{t=0}^{\infty}$ by the solution to the bargaining problem, (8), that is, $x_t = \ell_t = v(y_t) = \min\{v(y^*), d_t\}$ for each t. It remains to show that $\{(y_t, x_t, \ell_t)\}_{t=0}^{\infty}$ is the outcome of a credit equilibrium, (s^b, s^s), with buyer repayment strategy consistent with $\{d_t\}_{t=0}^{\infty}$. As in the proof of Proposition 1, the strategy follows a simple finite automaton with two states, $\chi_{i,t} \in \{G, A\}$, and each buyer's initial state is $\chi_{i,0} = G$. The law of motion of the buyer i's state are given by:

$$\chi_{i,t+1}\left[(\ell', x', i), \chi_{i,t}\right] = \begin{cases} A & \text{if } x' < \min(d_t, \ell') \text{ or } \chi_{i,t} = A \\ G & \text{otherwise} \end{cases}. \tag{44}$$

This law of motion is the same as (43), where d_t replaces x_t. The strategies are analogous to those constructed in the proof of Proposition 1, but with d_t as the maximum amount of debt the buyer repays: at date t, the buyer offers (y_t, ℓ_t) in state G, the seller accepts the offer (y', ℓ') iff $v(y') \leq \ell' \leq d_t$ and buyer state is G, and the buyer repays $\min(\ell', d_t)$ in the CM in state G if ℓ' is the loan issued in DM. Following exactly the same logic as in the proof of Proposition 1, (9) and (10) ensure that (s^b, s^s) is a credit equilibrium.

($\Rightarrow$) Here we show necessity. Let $\{d_t\}_{t=0}^{\infty}$ be a sequence consistent with a credit equilibrium outcome, $\{(y_t, x_t, \ell_t)\}_{t=0}^{\infty}$. By definition, $\{d_t\}_{t=0}^{\infty}$ satisfies (10). To show (9), consider a buyer at period-t CM with a loan size $\ell' = d_t$ (perhaps on an off-equilibrium path). For repayment of d_t to be optimal in state G, (9) must hold, i.e., the buyer prefers repaying d_t to permanent autarky.

Proof of Corollary 2 Rewrite the incentive-compatibility constraint (11) at time $t+1$ and multiply it by β to obtain:

$$\beta d_{t+1} \leq \beta^2 \left\{\alpha \left[u(y_{t+2}) - v(y_{t+2})\right] + d_{t+2}\right\}. \tag{45}$$

Combining (11) and (45) we get:

$$d_t \leq \beta \left\{ \alpha \left[u(y_{t+1}) - v(y_{t+1}) \right] \right\} + \beta^2 \left\{ \alpha \left[u(y_{t+2}) - v(y_{t+2}) \right] \right\} + \beta^2 d_{t+2}.$$

By successive iterations we generalize the inequality above as follows:

$$d_t \leq \sum_{s=1}^{T} \beta^s \left\{ \alpha \left[u(y_{t+s}) - v(y_{t+s}) \right] \right\} + \beta^{t+T} d_{t+T}. \tag{46}$$

Since by assumption, $\{d_t\}$ is bounded, $\lim_{T\to\infty} \beta^{t+T} d_{t+T} = 0$. Hence, by taking T to infinity, it follows from (46) that $\{d_t\}$ satisfies (9).

Proof of Proposition 2 Define the right side of (12) as a function

$$\Psi(d) = \alpha \left\{ u\left[z(d) \right] - v\left[z(d) \right] \right\}. \tag{47}$$

Ψ is continuous in d with $\Psi(0) = 0$ and $\Psi(d) = \alpha \left[u\left(y^*\right) - v\left(y^*\right) \right]$ for all $d \geq v(y^*)$. Moreover, it is differentiable with

$$\Psi'(d) = \alpha \left\{ \frac{u'[z(d)] - v'[z(d)]}{v'[z(d)]} \right\} \text{ if } d \in (0, v(y^*)], \text{ and } \Psi'(d) = 0 \text{ if } d > v(y^*).$$

This derivative is decreasing in d for all $d \in (0, v(y^*))$. Hence, Ψ is a concave function of d, and the set of values for d that satisfies (12) is an interval $[0, d^{\max}]$, where $d^{\max} \geq 0$ is the largest number that satisfies $\Psi(d^{\max}) = rd^{\max}$. Moreover, $d^{\max} > 0$ if and only if $\Psi'(0) > r$, which is always satisfied since $\Psi'(0) = \infty$ by assumption on preferences.

Proof of Lemma 1 Define the correspondence $\Gamma : \mathbb{R}_+ \supset \mathbb{R}_+$ as follows:

$$\Gamma(d) = \left\{ x \in \mathbb{R}_+ : r\left(1+\beta\right)x \leq \alpha \left\{ u\left[z(d)\right] - v\left[z(d)\right] \right\} + \beta\alpha \left\{ u\left[z(x)\right] - v\left[z(x)\right] \right\} \right.. \tag{48}$$

Then, $\gamma(d) = \max \Gamma(d)$. First we show that $\Gamma(d)$ is a closed interval and γ is well-defined. By definition, $x \in \Gamma(d)$ if and only if

$$r(1+\beta)x \leq \Psi(d) + \beta\Psi(x),$$

where $\Psi(d) = \alpha \left\{ u\left[z(d)\right] - v\left[z(d)\right] \right\}$. Using a similar argument to that in Proposition 2, $\Gamma(d)$ is a closed interval with zero as the lower end point. Thus, γ is well-defined, and $\gamma(d)$ is the largest x that satisfies

$$r\left(1+\beta\right)x = \Psi(d) + \beta\Psi(x). \tag{49}$$

Moreover, if $d > d'$, then $\Gamma(d') \subseteq \Gamma(d)$, and hence γ is a non-decreasing function: Because $\Psi(d)$ is constant for all $d \geq v(y^*)$, γ is constant for all $d \geq v(y^*)$, but it is strictly increasing for $d < v(y^*)$. Now we show that

$$\gamma(0) > 0, \;\; \gamma(d^{\max}) = d^{\max},$$

where $d^{\max}$ is given in Proposition 2. First, as $\Psi(0) = 0$, and $\Psi(x)$ is a concave function, $\gamma(0) > 0$ if and only if $r(1+\beta) < \Psi'(0) = \infty$, which holds by Inada conditions. Moreover, as the two curves $r(1+\beta)x$ and $\beta\Psi(x)$ intersects at $\gamma(0) \equiv d^{\min} > 0$, by concavity of Ψ we have $\beta\Psi'(d^{\min}) < r(1+\beta)$. Second, by Proposition 2, $d^{\max} > 0$ and $rd^{\max} = \Psi(d^{\max})$. Therefore, $r(1+\beta)d^{\max} = \Psi(d^{\max}) + \beta\Psi(d^{\max})$ and hence $\gamma(d^{\max}) = d^{\max}$.

Finally, we show that γ is a concave function. Applying the implicit function theorem to (49), for all $0 < d < v(y^*)$,

$$\gamma'(d) = \frac{\Psi'(d)}{(1+\beta)r - \beta\Psi'[\gamma(d)]}.$$

Note that $(1+\beta)r - \beta\Psi'[\gamma(0)] = (1+\beta)r - \beta\Psi'(d^{\min}) > 0$ and hence $(1+\beta)r - \beta\Psi'[\gamma(d)] > 0$ for all d. By concavity of Ψ, $\gamma'(d)$ is decreasing in d. Hence, γ is a concave function.

Proof of Proposition 3 Notice that, by definition, any pair (d_0, d_1) that satisfies $d_0 \leq \gamma(d_1)$ and $d_1 \leq \gamma(d_0)$ also satisfies (13) with $y_0 = z(d_0)$ and $y_1 = z(d_1)$, and hence (d_0, d_1) is a 2-period credit cycle. By Lemma 1, γ is a concave function with $\gamma(0) > 0$ and $\gamma(d^{\max}) = d^{\max}$, and hence, $\gamma(d) > d$ for all $d \in [0, d^{\max})$, where $d^{\max}$ is given in Proposition 2. Thus, for each $d_0 \in [0, d^{\max})$, the interval $[d_0, \gamma(d_0)]$ is nondegenerate and $\gamma(d_0) < d^{\max}$. Hence, for each $d_1 \in [d_0, \gamma(d_0)]$, $d_0 \leq d_1 < \gamma(d_1)$, where we used that $\gamma(d_0) > d$ for all $\gamma(d_0) < d^{\max} < d^{\max}$, so (d_0, d_1) is a 2-period credit cycle. This gives a full characterization of the set of 2-period cycles with $d_0 \leq d_1$, and the set of cycles with $d_1 \leq d_0$ is its mirror image with respect to the 45° line. Thus, for each $d_0 \in [0, d^{\max})$, the set $\Omega(d_0)$ is a nondegenerate interval.

Proof of Corollary 3 As shown earlier, a pair (d_0, d_1) is a 2-period cycle if and only if $d_0 \leq \gamma(d_1; \alpha, r)$ and $d_1 \leq \gamma(d_0; \alpha, r)$, where γ is given by Lemma 1. Note that here we make the parameters (α, r) explicit. By Proposition 3, for all $d_0 \in [0, d^{\max})$, there exists a continuum of d_1 such that (d_0, d_1) is a 2-period cycle. Now, $d_1 \leq \gamma(d_0; \alpha, r)$ if and only if

$$\frac{r(2+r)}{1+r} d_1 \leq \Psi(d_0; \alpha) + \frac{1}{1+r}\Psi(d_1; \alpha), \tag{50}$$

where $\Psi(d; \alpha) = \alpha\{u[z(d)] - v[z(d)]\}$. By the proof of Proposition 3, for each $d_0 \in [0, d^{\max})$, (d_0, d_1) is a 2-period cycle with $d_0 \leq d_1$ if and only if d_1 satisfies (50). Let $\overline{\Omega}(d_0; \alpha, r)$ be the set of such d_1. Because of symmetry between d_0 and d_1 in a 2-period cycle, it suffices to show that $\overline{\Omega}(d_0; \alpha, r)$ expands as α increases and as r decreases. Because $\Psi(d; \alpha)$ is strictly increasing in α, it follows that for any $\alpha' > \alpha''$, $d_1 \in \overline{\Omega}(d_0; \alpha'', r)$ implies that $d_1 \in \overline{\Omega}(d_0; \alpha', r)$, but there exists $d_1 \in \overline{\Omega}(d_0; \alpha', r)$ that is not in $\overline{\Omega}(d_0; \alpha'', r)$, that is, $\overline{\Omega}(d_0; \alpha'', r) \subsetneqq \overline{\Omega}(d_0; \alpha', r)$. Similarly, because the left-side of (50) is increasing in r but the right-side is decreasing in r, for any $r' > r''$, $d_1 \in \overline{\Omega}(d_0; \alpha, r')$ implies that $d_1 \in \overline{\Omega}(d_0; \alpha, r'')$, but there exists $d_1 \in \overline{\Omega}(d_0; \alpha, r'')$ that is not in $\overline{\Omega}(d_0; \alpha, r')$, that is, $\overline{\Omega}(d_0; \alpha, r') \subsetneqq \overline{\Omega}(d_0; \alpha, r'')$.

Proof of Corollary 4 Note that $d^{\max} \leq \upsilon(y^*)$ if and only if $r \geq \alpha \left[u(y^*) - \upsilon(y^*)\right] / \upsilon(y^*)$. We prove the two cases separately.

Case 1: $d^{\max} \leq \upsilon(y^*)$. Consider a 2-period cycle, (d_0, d_1), with $d_0 \leq d_1$. By Proposition 3, $d_0 \leq d_1 \leq \gamma(d_0) \leq d^{\max} \leq \upsilon(y^*)$. Thus, by (8), the loan contract is given by $\ell_t = d_t$ for $t = 0, 1$. The case $d_0 > d_1$ is completely symmetric.

Case 2: $d^{\max} > \upsilon(y^*)$. Consider a 2-period cycle, (d_0, d_1), with $d_0 \leq d_1$. If $d_0 \leq d_1 \leq \upsilon(y^*)$, then using identical arguments as case (1) above, we can show that the borrowing constraints always bind. Because, as shown in Lemma 1, $\gamma[\upsilon(y^*)] = d^{\max} > \upsilon(y^*)$, there exists a unique $\widehat{d}_0 < \upsilon(y^*)$ such that $\gamma(\widehat{d}_0) = \upsilon(y^*)$. Then, for any $d_0 \in (\widehat{d}_0, d^{\max}]$ and for any $d_1 \in [d_0, \gamma(d_0)]$, $d_1 > \upsilon(y^*)$, and hence, by (8), $\ell_1 = \upsilon(y^*) < d_1$. Thus, for any 2-period cycle, (d_0, d_1), with $\widehat{d}_0 < d_0 \leq \upsilon(y^*)$ and $d_1 \in [d_0, \gamma(d_0)]$, the borrowing constraint is slack in odd periods but binds in even periods. For any 2-period cycle, (d_0, d_1), with $\upsilon(y^*) < d_0$ and $d_1 \in [d_0, \gamma(d_0)]$, the borrowing constraints are slack in all periods. The case $d_0 > d_1$ is symmetric.

Proof of Proposition 4 By a similar argument to Proposition , a T-tuple, $(d_0, ..., d_{T-1}) \in \mathbb{R}_+^T$, is a T-period credit cycle if and only if $d_t \leq \gamma_T(d_{t+1}, .., d_{t+T-1})$, where $\gamma_T(d_0, ..., d_{T-2})$ is the largest x that satisfies

$$r\frac{\beta}{1-\beta^T}x = \sum_{t=0}^{T-2} \beta^t \Psi(d_t) + \beta^{T-1}\Psi(x), \tag{51}$$

and $\Psi(x) = \alpha \left\{u\left[z(x)\right] - \upsilon\left[z(x)\right]\right\}$. The function $\gamma_T(d_0, ..., d_{T-2})$ is a non-decreasing function in all its arguments. By similar arguments to Lemma 1, we can also show that $\gamma_T(d^{\max}, ..., d^{\max}) = d^{\max}$ and $\gamma_T(d, ..., d) > d$ for all $d \in [0, d^{\max})$. Moreover, γ_T is a concave function. For each $d_0 \in [0, d^{\max})$, define

$$\Omega_T(d_0) = \left\{(d_1, ..., d_{T-1}) \in \mathbb{R}_+^{T-1} : d_t \leq \gamma_T(d_{t+1}, .., d_{t+T-1}) \text{ for all } t = 0, ..., T\right\}.$$

The set $\Omega_T(d_0)$ is closed and bounded. Moreover, $(d_0, ..., d_0) \in \Omega_T(d_0)$ where all inequalities in the definition above are strict inequalities. Hence, there exists an open ball with a positive radius centered at $(d_0, ..., d_0)$ that is contained in $\Omega_T(d_0)$. Thus, $\Omega_T(d_0)$ has positive Lebesgue measure in $\mathbb{R}^{T-1}$. Finally, because γ_T is concave, $\Omega_T(d_0)$ is a convex set.

Proof of Proposition 5 Replace $d_t = \phi_t$ into the buyer's optimality condition in a monetary economy, (18), to get

$$d_t = \beta d_{t+1}\left[1+\alpha\frac{u'(y_{t+1})-\upsilon'(y_{t+1})}{\upsilon'(y_{t+1})}\right]. \tag{52}$$

The right side of (52), $[u'(y_{t+1})-\upsilon'(y_{t+1})]/\upsilon'(y_{t+1})$, is the derivative of the function, $u[\upsilon^{-1}(d_{t+1})]-d_{t+1}$, with respect to d_{t+1}. From the strict concavity of the function and the fact that it is equal to 0 when evaluated at $d_{t+1}=0$,

$$\frac{u'(y_{t+1})-\upsilon'(y_{t+1})}{\upsilon'(y_{t+1})}d_{t+1} < u(y_{t+1})-\upsilon(y_{t+1}). \tag{53}$$

From (52) and (53),

$$d_t < \beta\alpha\left[u(y_{t+1})-\upsilon(y_{t+1})\right]+\beta d_{t+1}. \tag{54}$$

Iterating (54),

$$d_t < \sum_{j=1}^{J}\beta^j\alpha\left[u(y_{t+j})-\upsilon(y_{t+j})\right]+\beta^J d_{t+J}. \tag{55}$$

Applying the transversality condition, $\lim_{J\to\infty}\beta^J d_{t+J}=0$ to (55), we prove that the sequence, $\{d_t\}$, solution to (52) satisfies (9), and hence it is part of a credit equilibrium.

Proof of Proposition 6 Here we show that for any distribution over $\mathbb{X}$ with a full support, denoted by π, we have a continuum of sunspot equilibria indexed by $d\in(0,d^{\max})$. For any $d\in(0,d^{\max})$, we have

$$rd < \alpha\{u[z(d)]-\upsilon[z(d)]\}. \tag{56}$$

Fix an element $\chi_0\in\mathbb{X}$ and let $\mathbb{X}_{-0}=\mathbb{X}-\{\chi_0\}$. Define the set

$$\Omega_{(\mathbb{X},\pi)}(d_{\chi_0}) = \left\{\langle d_\chi;\chi\in\mathbb{X}_{-0}\rangle : rd_\chi \le \pi(\chi_0)\alpha\{u[z(d_{\chi_0})]-\upsilon[z(d_{\chi_0})]\}+\sum_{\chi\in\mathbb{X}_{-0}}\pi(\chi)\alpha\{u[z(d_\chi)]-\upsilon[z(d_\chi)]\} \text{ for all } \chi\in\mathbb{X}\right\}.$$

By (56), the sequence $\langle d_\chi;\chi\in\mathbb{X}_{-0}\rangle$ with $d_\chi=d_{\chi_0}$ for all $\chi\in\mathbb{X}_{-0}$ is in $\Omega_{(\mathbb{X},\pi)}(d_{\chi_0})$ where all inequalities in the definition above are strict inequalities. Thus, the set $\Omega_{(\mathbb{X},\pi)}(d_{\chi_0})$ contains an open ball with a positive radius centered at $\langle d_\chi;\chi\in\mathbb{X}_{-0}\rangle$ with $d_\chi=d_{\chi_0}$ for all $\chi\in\mathbb{X}_{-0}$. Hence, it has a positive Lebesgue measure in $\mathbb{R}^{|\mathbb{X}_{-0}|}$ and almost all points in it satisfies $d_\chi\neq d_{\chi'}$ for all $\chi\neq\chi'$. Note that for any $\langle d_\chi;\chi\in\mathbb{X}_{-0}\rangle\in\Phi(d)$, $\langle d_\chi;\chi\in\mathbb{X}\rangle$ with $d_{\chi_0}=d$ is a sunspot credit equilibrium by (21).

Proof of Proposition 7 In the main text we have shown that (22) is necessary for the buyer to deliver their promised output in the DM, and, by (25), for a given sequence of debt limits, $\{d_t\}$, the equilibrium amont of loan is given by $\ell_t=\eta[z(d_t)]$ and hence $u(y_t)-\ell_t=\theta[u(y_t)-\upsilon(y_t)]$. Thus, (22) becomes (26). This proves the necessity. For sufficiency, let $\{(y_t,\ell_t,d_t)\}$ be a sequence satsifying (25) and (26). We use the same strategies as in the proof of Corollary 1, but we have to modify s_2^b in accordance with the new environment. As the buyer makes the production decision in the DM, the buyer strategy s_2^b becomes a delivery stratey that specifies the amount of the output that the buyer delivers to the seller in the CM, the difference being what the buyer consumes. Analogous to the strategies in Corollary 1, s_2^b satisfies the following threshold property. If ℓ' is the amount of output that the buyer promises to deliver in the period-t DM, then his actual delivery is $x' = \min\{\ell', d_t\}$. As in the proof of Corollary 1, (26) ensures that this delivery strategy is optimal.

Proof of Proposition 8 Form (26), a pair (d_0, d_1) is a 2-period cycle credit equilibirum outcome if and only if $d_0 \leq \gamma(d_1)$ and $d_1 \leq \gamma(d_0)$, where $\gamma(d)$ is the largest x that satisfies

$$r\lambda\,(1+\beta)\,x = \Psi(d) + \beta\Psi(x), \tag{57}$$

and

$$\Psi(d) = \alpha\theta\left\{u\left[z(d)\right] - v\left[z(d)\right]\right\}.$$

The left side of the equation in (57), $r\lambda\,(1+\beta)\,x$, is linear and increasing while the right side, $\beta\alpha\theta\left\{u\left[z(x)\right] - v\left[z(x)\right]\right\}$, is non-decreasing and concave. Given that the first term on the right side is non-negative, $\gamma(d)$ is well-defined. Note that $\gamma(d^{\max}) = d^{\max}$, where $d^{\max}$ is defined as the highest solution to $r\lambda d = \Psi(d)$, and, by similar arguments to Lemma 1, we can show γ is concave.

Note that $d^{\max} > 0$ if and only if $\Psi'(0) > r\lambda$. Now,

$$\Psi'(d) = \alpha\theta\left[\frac{u'[z(d)] - v'[z(d)]}{(1-\theta)u'[z(d)] + \theta v'[z(d)]}\right]$$

and hence $\Psi'(0) = \alpha\theta/(1-\theta)$, which shows that $d^{\max} > 0$ if and only if $r\lambda < \alpha\theta/(1-\theta)$.

When $d^{\max} > 0$, i.e., when $r\lambda < \alpha\theta/(1-\theta)$, for each $0 < d_0 < d^{\max}$, $d_0 < \gamma(d_0)$, and, for each $d_1 \in (d_0, \gamma(d_0)]$, $d_0 < \gamma(d_0) \leq \gamma(d_1)$. So any such (d_0, d_1) is a 2-period cycle and there are continuum of them.

To show the existence of 2-period cycles with periodic credit shutdowns, we need to show that $\gamma(0) > 0$. From (57), $\gamma(0) > 0$ if and only if $r\lambda(1+\beta) < \beta\Psi'(0) = \beta\alpha\theta/(1-\theta)$, which corresponds to the condition $r < \sqrt{1 + \alpha\theta/[\lambda(1-\theta)]} - 1$. Given that $\gamma(0) > 0$, any $(d_0, d_1) \in \{0\} \times (0, \gamma(0))$ is a credit equilibrium where credit shuts down in even periods.

Proof of Proposition 9 (1) Suppose that $y^* \leq y^{\max}$. Then, the outcome $\{(y_t, \ell_t)\}_{t=0}^{\infty}$ with $y_t = y^*$ and $\ell_t = v(y_t)$ for all t is implementable.

(2) Suppose that $y^* > y^{\max}$. We show that the optimal sequence that has $y_t = y^{\max}$ and $\ell_t = v(y_t)$ for all t. Suppose, by contradiction, that there is another sequence $\{y_t', \ell_t'\}_{t=0}^{\infty}$ satisfying (32) and (33) with

a strictly higher welfare. It then follows that $y^* \geq y'_t > y^{\max}$ for some t. Let t_0 be the first t such that $u(y'_t) - v(y'_t) > u(y^{\max}) - v(y^{\max})$. Now we show that for some $t_1 > t_0$, $y'_{t_1} > y'_{t_0}$. Suppose, by contradiction, that $y'_t \leq y'_{t_0}$ for all $t > t_0$. We have the following inequality,

$$v(y'_{t_0}) \leq \ell'_{t_0} \leq \lambda^{-1} \sum_{s=1}^{+\infty} \beta^s \alpha \left[u(y'_{t_0+s}) - \ell'_{t_0+s} \right] \leq \lambda^{-1} \sum_{s=1}^{+\infty} \beta^s \alpha \left[u(y'_{t_0}) - v(y'_{t_0}) \right],$$

where the first inequality follows from the seller participation constraint, (33), at $t = t_0$, the second follows from the buyer participation constraint, (32), and the third follows from $u(y'_{t_0+s}) - \ell'_{t_0+s} \leq u(y'_{t_0+s}) - v(y'_{t_0+s}) \leq u(y'_{t_0}) - v(y'_{t_0})$ since $u - v$ is increasing for $y < y^*$ and $\ell'_{t_0+s} \leq v(y'_{t_0+s})$ for all s. Because $y^{\max}$ is the maximal value of y'_{t_0} that equalizes the left side and the right side of this series of inequalities, it follows that $y'_{t_0} \leq y^{\max}$, a contradiction. So $y^* \geq y'_{t_1} > y'_{t_0}$ for some t_1 (and we choose $t_1 > t_0$ to be the first index for this to happen). By induction, we can then find a subsequence $\{y'_{t_i}\}$ that is strictly increasing and is bounded from above. So there exists a limit $\widetilde{y} = \lim_{i \to \infty} y'_{t_i} > y^{\max}$. Hence, by monotonicity, we have for all i,

$$rv(y'_{t_i}) \leq r\ell_{t_i} \leq \frac{\alpha[u(\widetilde{y}) - v(\widetilde{y})]}{\lambda},$$

and, by taking i to infinity, we have

$$rv(\widetilde{y}) \leq \frac{\alpha[u(\widetilde{y}) - v(\widetilde{y})]}{\lambda}.$$

However, as explained above, this implies that $\widetilde{y} \leq y^{\max}$, and this leads to a contradiction.

Proof of Proposition 10 The program that selects the best PBE is

$$\max_{\{y_t\}_{t=0}^{\infty}} \sum_{t=0}^{+\infty} \beta^t \alpha \left[u(y_t) - v(y_t) \right] \tag{58}$$

$$\text{s.t.} \quad \lambda \eta(y_t) \leq \alpha \sum_{s=1}^{+\infty} \beta^s \left[u(y_{t+s}) - \eta(y_{t+s}) \right] \tag{59}$$

$$y_t \leq y^* \text{ for all } t = 0, 1, 2, ... \tag{60}$$

(1) Suppose that $y^* \leq y^{\max}$. Then, the outcome $\{y_t\}_{t=0}^{\infty}$ with $y_t = y^*$ for all t is implementable.

(2) Suppose that $y^* > y^{\max}$ but $y^{\max} \leq \hat{y}$. We show that outcome $\{y_t\}_{t=0}^{\infty}$ with outcome $y_t = y^{\max}$ for all t is the optimum. Suppose, by contradiction, that there is another outcome $\{y'_t\}_{t=0}^{\infty}$ satisfying (59) and (60) with a strictly higher welfare. First we show that $y'_t \leq \hat{y}$ for all t. Suppose, by contradiction, that there is a t such that $y'_t > \hat{y}$. Then, because $\hat{y} \geq y^{\max}$,

$$\lambda \eta(y'_t) > \lambda \eta(\hat{y}) \geq \sum_{s=1}^{\infty} \beta^s \alpha[u(\hat{y}) - \eta(\hat{y})] \geq \sum_{s=1}^{\infty} \beta^s \alpha[u(y'_{t+s}) - \eta(y'_{t+s})],$$

a contradiction to (59). Given that this alternative outcome can only lie in the range $[0, \hat{y}]$ and hence the trade surplus is increasing in the output, the rest of the arguments are exactly the same as those in the proof of Proposition 9.

Proof of Proposition 11 In the supplementary Appendix S4 we show that the constrained-efficient allocation, $\{x_t, y_t\}$, can be determined recursively as follows:

$$\begin{aligned} V(\omega) &= \max_{y,\omega'} \{\alpha [u(y) - v(y)] + \beta V(\omega')\} && (61) \\ \text{s.t.} \quad & -\eta(y) + \beta \frac{\omega'}{\lambda} \geq 0 && (62) \\ & \beta\omega' \geq \{\omega - \alpha [u(y) - \eta(y)]\} && (63) \\ & y \in [0, y^*], \ \omega' \in [0, \bar{\omega}], && (64) \end{aligned}$$

with $\omega_0 = 0$, $\omega_{t+1} = \omega'(\omega_t)$, $y_t = y(\omega_t)$, and $x_t = \eta(y_t)$. Moreover, the value function V is unique, and it is nonincreasing and concave.

The Lagrangian associated with the above Bellman equation is

$$\begin{aligned} \mathcal{L} &= \alpha [u(y) - v(y)] + \beta V(\omega') + \xi \left(\beta \frac{\omega'}{\lambda} - \eta(y)\right) \\ & \quad + \nu \{\alpha [u(y) - \eta(y)] + \beta\omega' - \omega\}, && (65) \end{aligned}$$

where the Lagrange multipliers, ξ and ν, are non-negative. In general V may not be differentiable everywhere. However, because V is concave, the following first-order conditions are still necessary and sufficient for (y, ω') to be optimal:

$$\alpha [u'(y) - v'(y)] - \xi\eta'(y) + \nu\alpha [u'(y) - \eta'(y)] = 0 \quad (66)$$

$$\beta V'_+(\omega') + \beta \frac{\xi}{\lambda} + \beta\nu \leq 0 \leq \beta V'_-(\omega') + \beta \frac{\xi}{\lambda} + \beta\nu, \quad (67)$$

where $V'_+(\omega') = \lim_{\omega \downarrow \omega'} V'(\omega)$ and $V'_-(\omega') = \lim_{\omega \uparrow \omega'} V'(\omega)$. Both $V'_+(\omega')$ and $V'_-(\omega')$ exist becuase of concavity. The envelope condition, provided that $V'(\omega)$ exists, is

$$V'(\omega) = \nu. \quad (68)$$

We define two critical values for the buyer's promised utility:

$$\omega^{\max} = \frac{\alpha [u(y^{\max}) - \eta(y^{\max})]}{1 - \beta} \text{ and } \bar{\omega} = \frac{\alpha[u(\hat{y}) - \eta(\hat{y})]}{1 - \beta}.$$

The first threshold is the buyer's life-time expected utility at the highest steady state, while the second is the maximum life-time expected utility achieved by the buyers across all steady states. Note that by the definition of $y^{\max}$, $\eta(y^{\max}) = \beta\omega^{\max}/\lambda$.

Part 1 The following claim provides conditions under which the constrained-efficient allocation corresponds to the highest steady state. In order to establish this claim, we shows that, for $\omega = 0$ and $\omega = \omega^{\max}$, the optimal solution to the maximizaton problem in (61)-(64) is $(\omega^{\max}, y^{\max})$.

Claim 1 *If $\hat{y} < y^{\max} < y^*$ and $\lambda \geq \alpha\left[1 - u'(y^{\max})/\eta'(y^{\max})\right]$, then the unique solution to (61)-(64) is*

$$V(\omega) = \frac{\alpha\left[u(y^{\max}) - v(y^{\max})\right]}{1-\beta} \text{ if } \omega \in [0, \omega^{\max}], \tag{69}$$

$$= \frac{\alpha\left\{u[g(\omega)] - v[g(\omega)]\right\}}{1-\beta} \text{ if } \omega \in (\omega^{\max}, \bar{\omega}], \tag{70}$$

where $g(\omega)$ is the unique solution to $\alpha[u(y) - \eta(y)] = (1-\beta)\omega$ for all $\omega \in (\omega^{\max}, \bar{\omega}]$.

The function V given by (69)-(70) is flat in the interval $[0, \omega^{\max}]$ and is strictly concave for all $\omega \in (\omega^{\max}, \bar{\omega})$, and hence is concave overall. To show the strict concavity, we compute $V''(\omega)$ for all $\omega \in (\omega^{\max}, \bar{\omega})$. By the Implicit Function Theorem, we have

$$g'(\omega) = \frac{1-\beta}{\alpha\{u'[g(\omega)] - \eta'[g(\omega)]\}} < 0,$$

and hence

$$V'(\omega) = \frac{u'[g(\omega)] - v'[g(\omega)]}{u'[g(\omega)] - \eta'[g(\omega)]} \tag{71}$$

for all $\omega \in (\omega^{\max}, \bar{\omega})$. Thus,

$$\begin{aligned} V''(\omega) &= \frac{\{u''[g(\omega)] - v''[g(\omega)]\}\{u'[g(\omega)] - \eta'[g(\omega)]\}}{\{u'[g(\omega)] - \eta'[g(\omega)]\}^2} g'(\omega) \\ &+ \frac{-\{u'[g(\omega)] - v'[g(\omega)]\}\{u''[g(\omega)] - \eta''[g(\omega)]\}}{\{u'[g(\omega)] - \eta'[g(\omega)]\}^2} g'(\omega) < 0. \end{aligned}$$

Note that, for all $\omega \in (\omega^{\max}, \bar{\omega})$, $u'[g(\omega)] - \eta'[g(\omega)] < 0$ as $g(\omega) > \hat{y}$ and that $u'[g(\omega)] - v'[g(\omega)] > 0$ as $g(\omega) \leq y^{\max} < y^*$.

To prove that V satisfies (69) and (70), we consider two cases.

(a) Suppose that $\omega \in [0, \omega^{\max}]$. The solution to the maximization problem in (61)-(64) is given by $(\omega', y) = (\omega^{\max}, y^{\max})$. This solution is feasible because $(\omega^{\max}, y^{\max})$ satisfies (63) for all $\omega \leq \omega^{\max}$ and it satisfies (62) at equality. Next, we show that it satisfies (66)-(67) with $\nu = 0$ and

$$\xi = \frac{\alpha[u'(y^{\max}) - v'(y^{\max})]}{\eta'(y^{\max})} > 0.$$

The condition (66) holds by the definition of ξ. To establish (67), first note that $V'_{-}(\omega^{\max}) = 0$ and

$$V'_{+}(\omega^{\max}) \equiv \lim_{\omega \downarrow \omega^{\max}} V'(\omega) = \frac{u'(y^{\max}) - v'(y^{\max})}{u'(y^{\max}) - \eta'(y^{\max})}.$$

Thus, $V'_{-}(\omega^{\max}) + \xi/\lambda > 0$ and the first inequality in (67) holds if and only if

$$\begin{aligned} & V'_{+}(\omega^{\max}) + \frac{\xi}{\lambda} \\ = & \frac{1}{\lambda}\frac{[u'(y^{\max}) - v'(y^{\max})]}{\eta'(y^{\max})}\left\{\alpha + \lambda\frac{\eta'(y^{\max})}{u'(y^{\max}) - \eta'(y^{\max})}\right\} \leq 0, \end{aligned}$$

and, because $\hat{y} < y^{\max} < y^*$ and hence $u'(y^{\max}) - v'(y^{\max}) > 0$ and $u'(y^{\max}) - \eta'(y^{\max}) < 0$, the last inequality holds if and only if

$$\alpha\left[\frac{u'(y^{\max})}{\eta'(y^{\max})} - 1\right] \geq -\lambda,$$

that is, $\lambda \geq \alpha\,[1 - u'(y^{\max})/\eta'(y^{\max})]$. This implies V satisfies (69).

(b) Suppose that $\omega \in (\omega^{\max}, \bar{\omega})$. Here we show that $(\omega', y) = (\omega, g(\omega))$ is the solution to the maximization problem in (61)-(64). This solution is feasible: (63) holds by construction; because $\omega' = \omega = \alpha[u(y) - \eta(y)]/(1-\beta)$ and because $y = g(\omega) \leq y^{\max}$,

$$\lambda\eta(y) \leq \beta\alpha[u(y) - \eta(y)]/(1-\beta),$$

(62) holds. Next, we show that the FOC's (66) and (67) are satisfied by $(\omega', y) = (\omega, g(\omega))$ with $\xi = 0$ and

$$\nu = -\frac{u'[g(\omega)] - v'[g(\omega)]}{u'[g(\omega)] - \eta'[g(\omega)]} > 0.$$

The FOC for y, (66), holds by the definition of ν. The FOC for ω', (67), holds if and only if

$$\nu + V'(\omega) = 0,$$

which holds by (71). Thus, if $\omega_0 = \omega \in (\omega^{\max}, \bar{\omega})$, then the optimal sequence is $(\omega_t, y_t) = (\omega, g(\omega))$ for all t. Hence, $V(\omega)$ is satisfies (70) for all $\omega \in (\omega^{\max}, \bar{\omega})$. Finally, V satisfies (70) at $\omega = \bar{\omega}$ by continuity.

Part 2 We will show that $V(\omega)$ has the same closed-form solution as derived in claim 1 when $\omega > \omega^{\max}$. Given this observation, we will establish that if $\omega = 0$ then $\omega' > \omega^{\max}$ and y can be solved in closed form.

Claim 2 *Suppose that $\hat{y} < y^{\max} < y^*$ and $\lambda < \alpha\,[1 - u'(y^{\max})/\eta'(y^{\max})]$. Then, there exists a unique (y_0, y_1) with $\hat{y} < y_1 < y^{\max} < y_0 < y^*$ that solves (41)-(42), and the unique V that solves (61)-(64) satisfies*

$$\begin{aligned} V(\omega) &= \alpha[u(y_0) - v(y_0)] + \frac{\beta}{1-\beta}\alpha[u(y_1) - v(y_1)] \quad \textit{if } \omega = 0, && (72) \\ &= \frac{\alpha}{1-\beta}\{u[g(\omega)] - v[g(\omega)]\} \qquad \textit{if } \omega \in [\omega^{\max}, \bar{\omega}], && (73) \end{aligned}$$

where $g(\omega)$ is given in Part 1.

The fact that V satisfies (73) follows the proof of the second case in the claim in the proof of Part 1 and the Contraction Mapping Theorem. Note that by (73), $V'(\omega)$ is given by (71) for $\omega > \omega^{\max}$ and hence the proof there applies exactly.

Here we show (72). First we rewrite the problem in (61)-(64) at $\omega = 0$ as follows:

$$\max_{y,\omega'} \left\{\alpha\left[u(y) - v(y)\right] + \beta V(\omega')\right\} \tag{74}$$

$$\text{s.t.} \quad -\eta(y) + \beta\frac{\omega'}{\lambda} \geq 0 \tag{75}$$

$$y \in [0, y^*],\ \omega' \in [0, \bar{\omega}]\,. \tag{76}$$

Note that (63) is trivially satisfied when $\omega = 0$. Now, conjecturing that $\omega' \geq \omega^{\max}$, we can replace $V(\omega')$ by the expression given by (73), y by y_0 and $g(\omega')$ by y_1, and transform the above problem to

$$\max_{(y_0,y_1)\in[0,y^*]\times[\hat{y},y^{\max}]} \left\{\alpha\left[u(y_0) - v(y_0)\right] + \alpha\frac{u(y_1) - v(y_1)}{r}\right\} \tag{77}$$

$$\text{s.t.} \qquad -\eta(y_0) + \alpha\frac{u(y_1) - \eta(y_1)}{\lambda r} \geq 0, \tag{78}$$

which is exactly the same as (41)-(42). By the Kuhn-Tucker conditions, a pair (y_0, y_1) solves the above problem if it satisfies the following FOC and feasibility condition:

$$\frac{u'(y_0) - v'(y_0)}{\eta'(y_0)} = -\frac{\lambda}{\alpha}\left[\frac{u'(y_1) - v'(y_1)}{u'(y_1) - \eta'(y_1)}\right] \tag{79}$$

$$\alpha[u(y_1) - \eta(y_1)] = r\lambda\eta(y_0). \tag{80}$$

In order to show that the solution (y_0, y_1) is also a solution to the problem in (61)-(64) at $\omega = 0$ we only need to verify our conjecture,

$$\omega_1 = \frac{1}{1-\beta}\alpha[u(y_1) - \eta(y_1)] > \omega^{\max},$$

because the necessary conditions, (79)-(80), are also sufficient by the concavity of V over its entire domain.

Now we show that there exists a unique pair (y_0, y_1) with $\hat{y} < y_1 < y^{\max} < y_0 < y^*$ that satisfies (79)-(80). For each $y_1 \in (\hat{y}, y^{\max}]$, define

$$h(y_1) = \eta^{-1}\left[\frac{\alpha}{r\lambda}[u(y_1) - \eta(y_1)]\right].$$

as the unique solution of y_0 to (80) for a given y_1. Note that $h(y^{\max}) = y^{\max}$. For any $y_1 \in (\hat{y}, y^{\max}]$,

$$h'(y_1) = \frac{\alpha}{r\lambda}\frac{[u'(y_1) - \eta'(y_1)]}{\eta'[h(y_1)]} < 0.$$

Substituting y_0 by its expression given by $h(y_1)$ in the left side of (79), we rewrite (79) as $H(y_1) = 0$ where

$$H(y_1) = \frac{u'[h(y_1)] - v'[h(y_1)]}{\eta'[h(y_1)]} + \frac{\lambda}{\alpha}\left[\frac{u'(y_1) - v'(y_1)}{u'(y_1) - \eta'(y_1)}\right].$$

The function $H(y_1)$ is continuous and strictly increasing in $(\hat{y}, y^{\max}]$ with

$$\lim_{y_1 \downarrow \hat{y}} H(y_1) = -\infty,$$

and, at $y_1 = y^{\max}$, we have

$$\begin{aligned} H(y^{\max}) &= \frac{u'(y^{\max}) - v'(y^{\max})}{\eta'(y^{\max})} + \frac{\lambda}{\alpha}\left[\frac{u'(y^{\max}) - v'(y^{\max})}{u'(y^{\max}) - \eta'(y^{\max})}\right] \\ &= [u'(y^{\max}) - v'(y^{\max})]\left\{\frac{1}{\eta'(y^{\max})} + \frac{\lambda}{\alpha}\left[\frac{1}{u'(y^{\max}) - \eta'(y^{\max})}\right]\right\} > 0 \end{aligned}$$

because $\lambda < \alpha\,[1 - u'(y^{\max})/\eta'(y^{\max})]$. Thus, by Intermediate Value Theorem, there exists a unique $y_1 \in (\hat{y}, y^{\max})$ such that $H(y_1) = 0$ and hence (79) holds for $(h(y_1), y_1)$, and $h(y_1) > y^{\max}$ as h is strictly decreasing with $h(y^{\max}) = y^{\max}$. This proves that there exists a unique pair (y_0, y_1) with $\hat{y} < y_1 < y^{\max} < y_0 < y^*$ that satisfies (79) and (80).

Finally, because $\hat{y} < y_1 < y^{\max} < y_0 < y^*$ and because $(\omega', y) = (\omega_1, y_0)$ with $\omega_1 = \alpha[u(y_1) - \eta(y_1)]/(1-\beta)$ is the solution to the maximization problem in (61)-(64) for $\omega = 0$, V satisfies (72). $\square$

SUPPLEMENTARY APPENDICES

S1. Equivalence between monetary and credit equilibria

Here we extend the equivalence result, Proposition 5, to other trading mechanisms. We first consider bargaining in the pairwise meetings and then consider Walrasian pricing for large group meetings. We adopt the environment introduced in Section 4 without record-keeping. The monetary trades follow a similar pattern to that in Section 3.3: buyers who cannot commit to deliver goods in the CM use money to buy DM goods from sellers in the DM. They produce CM goods in the first stage of each period in order to sell them for money in the CM. Notice that the timing of producing CM goods (whether it takes place in the first or second stage of each period) for buyer's behavior because it is only incentive-feasible to sell these goods in the CM for money. Sellers use money obtained from DM sales to buy CM goods. Because $\lambda \leq 1$, buyers never produce CM goods for self-consumption. As a result, the parameter λ plays no role in monetary equilbria. So with no loss of generality we set $\lambda = 1$.

Bargaining Under a general bargaining solution represented by the function $\eta(y)$, the sequence for the values of money, $\{\phi_t\}$, solves

$$\max_{m \geq 0} \{\phi_t m + \beta\alpha [u(y_{t+1}) - \eta(y_{t+1})]\}$$

where $\phi_{t+1} m = \eta(y_{t+1})$ for all t. Replace $d_t = \phi_t$ for all t in the above problem and take the FOC, we obtain

$$d_t = \beta d_{t+1} \left\{ \alpha \left[\frac{u'(y_{t+1})}{\eta'(y_{t+1})} - 1 \right] + 1 \right\}, \tag{81}$$

where $\eta(y_t) = d_t$ for all t. In the credit economy, the debt limits, $\{d_t\}$, solves

$$d_t \leq \beta \{\alpha [u(y_{t+1}) - \eta(y_{t+1})] + d_{t+1}\}. \tag{82}$$

Because η is concave, $u \circ \eta^{-1}(d_t) - d_t$ is convave in terms of the value of money. The right side of (81), $[u'(y_{t+1}) - \eta'(y_{t+1})] / \eta'(y_{t+1})$, is the derivative of the function, $u[\eta^{-1}(d_{t+1})] - d_{t+1}$, with respect to d_{t+1}. From the strict concavity of the function and the fact that it is equal to 0 when evaluated at $d_{t+1} = 0$,

$$\frac{u'(y_{t+1}) - \eta'(y_{t+1})}{\eta'(y_{t+1})} d_{t+1} < u(y_{t+1}) - \eta(y_{t+1}). \tag{83}$$

From (81) and (83),

$$d_t < \beta\alpha [u(y_{t+1}) - \eta(y_{t+1})] + \beta d_{t+1}. \tag{84}$$

Iterating (84),

$$d_t < \sum_{j=1}^{J} \beta^j \alpha [u(y_{t+j}) - \eta(y_{t+j})] + \beta^J d_{t+J}. \tag{85}$$

Applying the transversality condition, $\lim_{J\to\infty}\beta^J d_{t+J}=0$ to (85), we prove that the sequence, $\{d_t\}$, solution to (81) satisfies (82), and hence it is part of a credit equilibrium.

This concavity of η is satisfied for the proportional bargaining solution and for the general Nash bargaining solution under the functional forms for u and v that guarantee the concavity of the buyer's surplus.

Walrasian pricing Suppose the DM is competitive and p_t denotes the price of DM goods in terms of CM goods. In a monetary economy the buyer chooses money holdings solution to:

$$\max_{m,y_{t+1}\geq 0}\left\{-\phi_t m+\beta\alpha\left[u(y_{t+1})-p_{t+1}y_{t+1}\right]+\beta\phi_{t+1}m\right\}, \tag{86}$$

where, $\phi_{t+1}m\geq p_{t+1}y_{t+1}$. The first-order condition for (86) is

$$\phi_t=\beta\phi_{t+1}\left\{\alpha\left[\frac{u'(y_{t+1})}{p_{t+1}}-1\right]+1\right\}.$$

From the seller's maximization problem, $p_{t+1}=v'(y_{t+1})$ so that $\{\phi_t\}$ solves

$$\phi_t=\beta\phi_{t+1}\left\{\alpha\left[\frac{u'(y_{t+1})}{v'(y_{t+1})}-1\right]+1\right\}. \tag{87}$$

It should be noticed that it is the same first-order difference equation than the one obtained under buyer's take-it-or-leave-it offer. Notice, using $\phi_{t+1}=v'(y_{t+1})y_{t+1}$ by market-clearing (i.e., $m=1$), that

$$\phi_{t+1}\left[\frac{u'(y_{t+1})}{v'(y_{t+1})}-1\right]=u'(y_{t+1})y_{t+1}-v'(y_{t+1})y_{t+1}<u(y_{t+1})-v'(y_{t+1})y_{t+1},$$

from the concavity of u. Recall that a sufficient condition for the debt limits to be a credit equilibrium is

$$d_t\leq\beta\left\{\alpha\left[u(y_{t+1})-v'(y_{t+1})y_{t+1}\right]+d_{t+1}\right\}.$$

This proves that the phase of the monetary equilibrium is located to the left of the phase line of the credit equilibrium. Hence, by the same reasoning as before, any outcome of the monetary economy is an outcome of the credit economy.

S2. Existence of 2-period cycles under alternative mechanisms

Walrasian pricing Under Walrasian pricing, $\eta(y) = v'(y)y$. Here we show existence of a continuum of 2-period cycles when $\eta(y)$ is convex. Recall that $z(d) = \min\{\eta^{-1}(d), y^*\}$. Let $d^{\max}$ be the unique positive solution to

$$r\lambda d = \alpha \{u [z(d)] - \eta [z(d)]\}. \tag{88}$$

Lemma 2 *Suppose that $\eta(y)$ is convex. For each $d_0 \in [0, d^{\max})$, there is a nondegenerate interval, $\Omega(d_0)$, such that for any $d_1 \in \Omega(d_0)$, (d_0, d_1) is a (strict) 2-period cycle.*

Proof. Because $\eta(y)$ is convex, there is a unique positive number, denoted $y^{\max}$, such that

$$r\lambda\eta(y^{\max}) = \alpha[u(y^{\max}) - \eta(y^{\max})].$$

It can be verified that that $d^{\max}$ is given by

$$d^{\max} = \begin{cases} \eta(y^{\max}) \text{ if } y^* \geq y^{\max} \\ \frac{\alpha\{u(y^*)-\eta(y^*)\}}{r\lambda} \text{ otherwise.} \end{cases}$$

Note that any $d \in [0, d^{\max}]$ corresponds to a steady-state equilibrium. Let us turn to 2-period cycles. A pair, (d_0, d_1), is a 2-period cycle if for $t = 0, 1$,

$$r\lambda d_t \leq \frac{\alpha \{u [z(d_{t+1})] - \eta [z(d_{t+1})]\} + \beta\alpha \{u [z(d_t)] - \eta [z(d_t)]\}}{1 + \beta}. \tag{89}$$

Hence,

$$\Omega(d_0) = \{d_1 \geq 0 : (d_0, d_1) \text{ satisfies (89)}\}.$$

For all $d \in [0, d^{\max})$, because $r\lambda d < \alpha \{u [z(d)] - \eta [z(d)]\}$, $(d_0, d_1) = (d, d)$ satisfies (89) with a strict inequality. Hence, by continuity, there is a nonempty open set contained in $\Omega(d)$. Moreover, because η is concave, the set $\Omega(d)$ is convex and hence is a nondegenerate interval. ■

Nash bargaining For all $y \leq y^*$, $u(y) - \eta(y) \geq \theta[u(y) - v(y)]$ and hence $\eta(y) \leq (1 - \theta)u(y) + \theta v(y)$. Under proportional bargaining a 2-period cycle solves

$$r\lambda [(1 - \theta)u(y_t) + \theta v(y_t)] \leq \frac{\{\alpha\theta [u(y_{t+1}) - v(y_{t+1})] + \beta\alpha\theta [u(y_t) - v(y_t)]\}}{1 + \beta}.$$

It implies

$$r\lambda\eta(y_t) \leq \frac{\{\alpha [u(y_{t+1}) - \eta(y_{t+1})] + \beta\alpha [u(y_t) - \eta(y_t)]\}}{1 + \beta}.$$

Hence (y_t, y_{t+1}), and the associated $(d_t, d_{t+1}) = (\eta(y_t), \eta(y_{t+1}))$, is a credit cycle under generalized Nash bargaining.

S3. Core and competitive equilibrium

Recall that an allocation, $\mathcal{L} = \{(y(i), x(i)), (y(j), x(j)) : i \in \mathbb{B}, j \in \mathbb{S}\}$, where $(y(i), x(i))$ denotes buyer i's DM and CM consumptions and $(y(j), x(j))$ denotes seller j's DM and CM consumptions, is in the core if there is no blocking (finite) coalition, $\mathcal{I} \subset \mathbb{B} \cup \mathbb{S}$, such that each agent in $\mathcal{I}$ enjoys at least the same utility as his allocation in $\mathcal{L}$, but at least one of them is strictly better off. Now we show that the only core allocation is the competitive outcome, with debt limit, d, is given by the symmetric allocation, (y, ℓ), such that $\ell = \eta(y) \equiv v'(y)y$ and $y = \min\{y^*, \eta^{-1}(d)\}$.

First notice that, by standard arguments, the comptitive outcome is in the core. For necessity, we restrict ourselves to symmetric allocations. For a justification of such assumption, see Mas-Colell et al. Note that to be in the core, $u(y) \geq \ell \geq v(y)$. First we show that $\ell = v'(y)y$. Suppose, by contradiction, $\ell \neq v'(y)y$. Assume that $\ell < v'(y)y$. The other direction has a similar proof. Let ε be so small that

$$[v'(y) - \varepsilon]y > \ell. \tag{90}$$

Consider a coalition with m buyers and n sellers such that with $\delta = m/n < 1$, we have

$$\frac{v(y) - v(\delta y)}{(1-\delta)y} > v'(y) - \varepsilon. \tag{91}$$

Consider the following allocation: each buyer consumes y and issues an IOU with face value ℓ, and each seller produces δy and receives an IOU with face value $\delta\ell$. Note that such allocation is feasible:

$$my = n\delta y \text{ and } m\ell = n\delta\ell.$$

Now, each buyer enjoys the same utility as before, but each seller has a higher utility: combining (90) and (91),

$$v(y) - v(\delta y) > [v'(y) - \varepsilon](1-\delta)y > (1-\delta)\ell,$$

and hence

$$\delta\ell - v(\delta y) > \ell - v(y).$$

This proves $\ell = v'(y)y = \eta(y)$. Finally, if $y < \min\{y^*, \eta^{-1}(d)\}$, then a buyer and a seller can form a coalition to increase surplus.

S4. Recursive formulation of the mechanism design problem

Here we show that we can solve the problem (58)-(60) recursively. First we show that recursive formulation with promised utility as a state variable is equivalent to the original sequence problem.

Lemma 3 *A sequence* $\{y_t\}_{t=0}^{\infty}$ *satisfies (59) and (60) if and only if there is a sequence* $\{\omega_t\}_{t=0}^{\infty}$ *such that, for all* $t = 0, 1, 2, ...,$

$$\omega_t \leq \alpha \left[u(y_t) - \eta(y_t)\right] + \beta \omega_{t+1}, \tag{92}$$

$$\eta(y_t) \leq \beta \omega_{t+1} / \lambda, \tag{93}$$

$$y_t \in [0, y^*], \tag{94}$$

$$\omega_t \in [0, \bar{\omega}]. \tag{95}$$

Proof. Suppose that $\{y_t\}_{t=0}^{\infty}$ satisfies (59) and (60). Then, define, for each $t = 0, 1, 2, ...,$

$$\omega_t = \sum_{s=0}^{\infty} \beta^s \alpha [u(y_{t+s}) - \eta(y_{t+s})]. \tag{96}$$

The right side of (59) is equal to $\beta \omega_{t+1} / \lambda$ for each t. Hence, $\{\omega_t, y_t\}_{t=0}^{\infty}$ satisfies (93). By definition of $\hat{y}$,

$$u(y_t) - \eta(y_t) \leq u(\hat{y}) - \eta(\hat{y}) \text{ for all } t \in \mathbb{N}_0.$$

It follows from (60) that $\{\omega_t\}_{t=0}^{\infty}$ satisfies (95). Finally, by (96),

$$\omega_t = \alpha[u(y_t) - \eta(y_t)] + \beta \sum_{s=0}^{\infty} \beta^s \alpha [u(y_{t+s+1}) - \eta(y_{t+s+1})] = \alpha[u(y_t) - \eta(y_t)] + \beta \omega_{t+1} \text{ for all } t \in \mathbb{N}_0.$$

Hence, $\{\omega_t, y_t\}_{t=0}^{\infty}$ satisfies (92).

Conversely, suppose that $\{\omega_t, y_t\}_{t=0}^{\infty}$ satisfies (92)-(95). Then, $\{y_t\}_{t=0}^{\infty}$ satisfies (60) by (94). To show (59), define, for each $t \in \mathbb{N}_0$,

$$\omega_t' = \sum_{s=0}^{\infty} \beta^s \alpha [u(y_{t+s}) - \eta(y_{t+s})]. \tag{97}$$

By (93), it suffices to show that $\omega_t \leq \omega_t'$ for all $t \geq 0$. Let t be given. We show by induction on T that

$$\omega_t \leq \sum_{s=0}^{T} \beta^s \alpha [u(y_{t+s}) - \eta(y_{t+s})] + \beta^{T+1} \omega_{T+1}. \tag{98}$$

When $T = 0$, this follows from (92). Suppose that it holds for T. Then,

$$\begin{aligned} \omega_t &\leq \sum_{s=0}^{T} \beta^s \alpha [u(y_{t+s}) - \eta(y_{t+s})] + \beta^{T+1} \omega_{T+1} \\ &= \sum_{s=0}^{T} \beta^s \alpha [u(y_{t+s}) - \eta(y_{t+s})] + \beta^{T+1} \left\{ \alpha \left[u(y_{T+1}) - \eta(y_{T+1}) \right] + \beta \omega_{T+2} \right\} \\ &= \sum_{s=0}^{T+1} \beta^s \alpha [u(y_{t+s}) - \eta(y_{t+s})] + \beta^{T+2} \omega_{T+2}. \end{aligned}$$

This proves (98). Now, because, by (95), $\omega_{T+1} \leq \bar{\omega}$ for all T, it follows from the limit by taking T to infinity in (98) that $\omega_t \leq \omega_t'$. ■

Because of Lemma 3, we may replace the constraints (59) and (60) by (92)-(95). Note that the initial condition for the promised utility, ω_0, is also a choice variable.

Define the planner's value function, $V(\omega)$, as follows:

$$V(\omega) = \max_{\{y_t\}_{t=0}^{\infty}} \sum_{t=0}^{+\infty} \beta^t \alpha \left[u(y_t) - \upsilon(y_t)\right]$$

subject to (92)-(95) with $\omega_0 = \omega$. From the Principle of Optimality V satisfies the following Bellman equations,

$$V(\omega) = \max_{y,\omega'} \{\alpha \left[u(y) - \upsilon(y)\right] + \beta V(\omega')\} \tag{99}$$

$$\text{s.t.} \quad -\eta(y) + \beta \frac{\omega'}{\lambda} \geq 0 \tag{100}$$

$$\beta\omega' \geq \{\omega - \alpha \left[u(y) - \eta(y)\right]\} \tag{101}$$

$$y \in [0, y^*], \ \omega' \in [0, \bar{\omega}]\,. \tag{102}$$

The proposition below shows that the above Bellman equation is well-defined and that V is uniquely determined. As a result, the maximization problem (58)-(60) is reduced to

$$\max_{\omega_0 \in [0,\bar{\omega}]} V(\omega_0).$$

Proposition 12 *Suppose that $y^* > y^{\max} > \hat{y}$.*

(1) The value function V is the unique solution to (99)-(102), and is continuous and weakly decreasing in ω.

(2) The function V is concave in ω if η is convex.

Proof. (1) First we show that, for any $\omega \in [0, \bar{\omega}]$, the set of elements $(y, \omega') \in [0, y^*] \times [0, \overline{\omega}]$ satisfying (100)-(102) is nonempty and hence the maximization problem is well-defined. For all $\omega \in [0, \bar{\omega}]$, define $y_\omega \leq \hat{y} \leq y^*$ as the unique solution to

$$\omega = \frac{\alpha}{1-\beta}[u(y_\omega) - \eta(y_\omega)]. \tag{103}$$

As $u(0) - \eta(0) = 0$ and $\frac{\alpha}{1-\beta}[u(\hat{y}) - \eta(\hat{y})] = \overline{\omega}$, such $y_\omega \in [0, \hat{y}]$ exists by the Intermediate Value Theorem. We claim that (y_ω, ω') satisfies (100)-(102) for any $\omega' \in [\omega, \bar{\omega}]$. First (102) holds by construction. Moreover, rearranging (103), we have

$$\beta\omega = \omega - \alpha[u(y_\omega) - \eta(y_\omega)]$$

which implies (101) for any $\omega' \geq \omega$. Finally, by (103) and the fact that $y \leq \hat{y} \leq y^{\max}$,

$$\eta(y_\omega) \leq \beta\frac{\omega}{\lambda} \leq \beta\frac{\omega'}{\lambda}$$

for any $\omega' \geq \omega$.

We now show that the Bellman equation (100)-(102) has a unique solution. Let $\mathcal{C}[0,\bar{\omega}]$ be the complete metric space of continuous functions over $[0,\bar{\omega}]$ equipped with the sup norm. Define $T : \mathcal{C}[0,\bar{\omega}] \to \mathcal{C}[0,\bar{\omega}]$ by

$$T(W)(\omega) = \max_{y,\omega'} \left\{\alpha\left[u(y) - v(y)\right] + \beta W(\omega')\right\},$$

subject to (100)-(102). Note that $T(W) \in \mathcal{C}[0,\bar{\omega}]$ by the Theorem of Maximum. The mapping T satisfies the Blackwell sufficient condition (Lucas and Stokey, 1989, Theorem 3.3), and hence T is a contraction mapping, which admits a unique fixed point by the Banach Fixed-Point Theorem. Hence, V is the unique solution to the Bellman equation and is continuous.

Notice that by decreasing ω we increase the set of (y,ω') that satisfies (100)-(102), but without affecting the objective function. Hence, V is weakly decreasing.

(2) Assume now that η is convex. To show that V is concave, we show that T preserves concavity. Let $\omega_0, \omega_1 \in [0,\bar{\omega}]$ be given. Let (y_0,ω_0) and (y_1,ω_1) solves (100)-(102) for ω_0 and ω_1, respectively. let $\epsilon \in (0,1)$ be given. Then,

$$\begin{aligned}
& T(W)(\epsilon\omega_0 + (1-\epsilon)\omega_1) \\
\geq\ & \alpha\left[u(\epsilon y_0 + (1-\epsilon)y_1) - v(\epsilon y_0 + (1-\epsilon)y_1)\right] + \beta W(\epsilon\omega_0' + (1-\epsilon)\omega_1') \\
\geq\ & \alpha\epsilon[u(y_0) - v(y_0)] + \alpha(1-\epsilon)[u(y_0) - v(y_0)] + \beta[\epsilon W(\omega_0') + (1-\epsilon)W(\omega_0')] \\
=\ & \epsilon T(W)(\omega_0) + (1-\epsilon)T(W)(\omega_1).
\end{aligned}$$

The first inequality follows from the fact that $(\epsilon y_0 + (1-\epsilon)y_1, \epsilon\omega_0' + (1-\epsilon)\omega_1')$ also satisfies (100)-(102) for $\omega = \epsilon\omega_0 + (1-\epsilon)\omega_1$ because η is convex. The second inequality follows from the concavity of $u - v$ and the assumed concavity of W. ■