Information aggregation in a large two-stage market game*

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September 6, 2013

Abstract

A market-game mechanism is studied in a two-divisible-good, pure-exchange setting with potential information-aggregation. The mechanism is simple: agents' actions are quantities and outcomes are determined by simple algorithms that do not depend on the details of the economy. There are two stages. First, agents make offers, which are provisional for all but a small, randomly selected group. Then, those offers are announced, and everyone else gets to make new offers with payoffs determined by a Shapley-Shubik market game. For a finite and large number of players, there exists an almost ex post efficient equilibrium. Conditions for uniqueness are provided. (99 words)

Key words: mechanism-design, information-aggregation, market-game, efficiency, rational-expectations equilibrium.

JEL classification numbers: D82, D43

1 Introduction

We study strategic trade in a two-divisible-good, pure-exchange setting with potential information aggregation. There is an unobserved state-of-the-world with dispersed and incomplete information about that state in the form of private signals. The realized utility of an agent depends both on the state and on the private signal received. This information-preference structure is borrowed from Reny and Perry [11] and Gul and Postlewaite [6]. The state may be interpreted as the qualities of the goods that affect all agents' utilities. In the language of *auction* theory, the model is a mixed *common-private value* setting. One special case is the pure private-value setting in which the state does not affect agents' utilities, but the private signals are correlated because their distribution is determined by the state (as in Cripps and Swinkels [2]). Another special case is the pure common-value

*We are indebted to Jim Jordan for very helpful conversations at the inception of this work.

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setting where the signals do not affect utilities. From a more general point of view, the presence of private signals makes it an *adverse-selection* model in the sense that after receiving a private signal, each agent knows something that others do not know. For such settings, a long-standing theoretical challenge is to devise mechanisms that have desirable properties. Here we consider two such properties: simplicity of the mechanism and efficiency of the mechanism.

There are two aspects of simplicity. The first is detail-freeness; a mechanism is detail-free if it does not rely on specific information about the economy, such as the functional form of agents' utilities or the way that private signals relate to the unobservable state.[1] The second is simplicity of both participants' actions and the mapping used to go from participants' actions to the outcome. As regards efficiency, we follow all the work on information aggregation and use a notion of ex post efficiency.

Some of the literature has focused solely on efficiency. Gul and Postlewaite [6] and McLean and Postlewaite [8] construct direct mechanisms that achieve almost ex post efficiency in environments with many divisible goods, environments that include ours as a special case. However, as is widely recognized, their direct mechanisms are not detail-free. Presumably, that is why McLean and Postlewaite [8] (pages 2,439 and 2,441) do not regard their mechanism as suitable for actual use.

Some of the literature has focused on both properties. Reny and Perry [11] and Vives [14] use double-auction mechanisms and achieve ex post efficiency in special, two-good, quasi-linear settings: Reny and Perry [11] assumes unit demands, while Vives [14] assumes quadratic utilities and normally distributed signals. However, except in special cases like those studied by Reny and Perry [11] and Vives [14], double auctions are not simple in terms of actions or the way the mechanism uses those actions. In order to achieve ex post efficiency in a general two-divisible-good setting, actions in a double auction would have to be general demand functions, which are not simple objects (see the remarks in Cripps and Swinkels [2]). Moreover, there is no simple algorithm that computes a market-clearing price in the double auction from arbitrary demand functions.

Here, we use a version of a *two-stage* market-game mechanism. As shown in previous work, *one-stage* market games do not, in general, aggregate information in a way that leads to efficiency because agents commit to quantities before the relevant information is revealed.[2] (In contrast, double auctions in which actions are demand functions permit an agent's realized trade to be contingent on the actions of others.) Our two-stage market-game mechanism deals with that failure by having some information revealed between stages and by having payoffs for almost all agents determined in the second stage. Despite those changes to the standard one-stage market-game mechanism, our mechanism retains its simplicity. First, it is detail-free; second, the actions at each stage are quantities (see, also, Dubey *et. al.* [5], page 108) and outcomes are derived from those actions using only

[1] Simplicity in this sense is part of the motivation in Hurwicz *et al.* [7].

[2] Palfrey [10] uses a Cournot mechanism and obtains ex post optimality, but, as Vives [13] points out, only because marginal cost is common and constant so that it does not matter how production is allocated among the firms in the model. Dubey *et al.* [5] study a dynamic market game with trades in multiple periods. They show that information may be aggregated, but only after trades and consumption at the first-period are observed. As they emphasize, this precludes ex post efficiency.

simple arithmetic operations.

Two- or multiple-stage mechanisms with information revealed between stages are common. A straw poll in a voting situation is one such mechanism. Another, which is closely related to our mechanism, is pari-mutuel betting. In pari-mutuel betting, running bet totals (and, therefore, odds) are announced before final odds are determined (via a market game).[3] Information aggregation has also been studied in experiments in which the focus is on how to design rewards in order to elicit what agents know (see, for example, Axelrod *et. al.* [1]).

Our mechanism elicits information at the first stage in the following way. At the first stage, each agent names an offer. Then, in a random fashion, the mechanism divides the agents into two groups: a small *inactive* group and a large *active* group. Those in the inactive group participate no further; their first-stage offers are executed at an exogenous price. Those in the active group participate in a second-stage market game after the histogram of their first-stage offers is announced—an announcement that mimics the announcement of running odds in pari-mutuel betting.

In our mechanism, an agent at the first-stage faces a trade-off. Contingent on becoming inactive, the agent's first-stage offer determines his final payoff so that it is in the agent's interest to reveal his private information. Contingent on becoming active, his first-stage offer affects his final payoff only by way of its influence on the beliefs of other active agents at the second stage. Therefore, second-stage beliefs, both on and off the equilibrium path, play a crucial role in determining how the first-stage action affects the payoff contingent on becoming active. As a consequence, those beliefs determine how the first-stage trade-off between the two contingent payoffs is resolved.

We have three main results—one about existence, one about uniqueness, and one about ex post optimality. All are related to ex post competitive equilibria of a corresponding limit economy—an economy in which the state is known and in which the fraction of agents with each realization of the private signal is the probability of receiving that signal conditional on the state. From the ex ante perspective, it is standard to label any such competitive equilibrium (CE) a rational-expectations CE (see, for example, Reny and Perry [11]). In this sense, our results establish a strategic foundation for rational-expectations CE in a new setting—one with divisible goods, general preferences, and a general information structure.

As regards existence, we construct a symmetric equilibrium in pure strategies in which stage-1 actions reveal the private-information held by all active agents through the announced histogram. We call such an equilibrium a *separating equilibrium.* Our first result shows that if some mild genericity condition holds and if the finite number of agents is sufficiently large, then there exists a separating equilibrium in which stage-2 behavior converges almost surely to a CE of the ex post limit economy conditional on each state.

Our existence proof has two main ingredients. The first ingredient shows existence of second-stage equilibria that converge almost surely to a given CE in the limit economy as the finite population gets large conditional on each state. This follows from a lower hemi-

[3] Versions of pari-mutuel betting have been used for more than a century.

continuity result that treats the signal-configuration as a parameter and implies existence of a second-stage equilibrium close to the given CE for any configuration close to the limit distribution implied by a given state. (We use that parametrization because we do not have deterministic replication of types.) The second ingredient is the formulation of beliefs so that, given the second-stage equilibrium that converges almost surely to a given CE conditional on each state, the first-stage trade-off is resolved entirely in favor of the payoff contingent on being inactive. Our formulation of beliefs is simple and plausible: it associates each (on- and off-equilibrium) offer with an agent type and then employs Bayes' rule to derive the beliefs over the state-of-the-world.

For uniqueness, we assume that the ex post limit economy has monotone competitive demand for each state-of-the-world—which, of course, implies a unique CE for each state. Then, under three mild additional assumptions, including a restriction on off-equilibrium beliefs that is similar to the "no-signaling-what-you-don't-know" restriction in Fudenberg and Tirole [4], the first-stage trade-off is resolved entirely in favor of the payoff contingent on being inactive in any symmetric equilibrium in pure strategies. As a consequence, under our genericity condition, any symmetric equilibrium in pure strategies is separating and has second-stage behavior that converges almost surely to the unique ex post CE of the limit economy conditional on each state.

Finally, we show that the separating equilibrium constructed in our existence result is *almost* ex post efficient, where the notion of efficiency is similar to that in Gul and Postlewaite [6] and in McLean and Postlewaite [8]. This efficiency result is new in the literature that focuses on detail-free mechanisms. Our result is for divisible goods and a general information structure, while Reny and Perry [11] has unit demands and Vives [14] has quadratic utilities and normally distributed signals.

2 The environment and the mechanism

Our economy is an endowment economy with two goods and N agents. (The set of agents is denoted $\mathcal{N}$.) Each agent is assigned a type, denoted x, where x is drawn from a finite set X. An agent of type $x \in X$ maximizes expected utility with ex post utility function, $u(q, r; x, z)$, where $(q, r) \in \mathbb{R}^2_+$ is the vector of quantities of the two goods consumed and z is drawn from Z, a finite set of states-of-the-world. The function $u(\cdot, \cdot; x, z)$ is strictly increasing, strictly concave, continuously twice differentiable, and satisfies Inada conditions. For simplicity, each agent is endowed with the per capita endowment of each good, denoted $(\bar{q}, \bar{r})$.

The sequence of events is as follows. First, nature draws a state-of-the-world $z \in Z$ with probability $\pi(z)$, a state which no one observes. Then each agent gets a type realization, $x \in X$, which is private to the agent. Conditional on the realization z, these realizations are *i.i.d.* across people. We denote the probability for an agent to be of type x conditional on state z by $\mu_z(x)$, and denote the implied posterior probability that the state is z conditional on signal x by $\tau_x(z)$. We assume that $\pi(z) > 0$ for each $z \in Z$ and that

$\mu_z(x) > 0$ for each $x \in X$ and $z \in Z$.[4] We also assume that x is informative in the sense that $z \neq z'$ implies $\mu_z(x) \neq \mu_{z'}(x)$ for some $x \in X$. (This informativeness assumption is without loss of generality: If $\mu_z(x) = \mu_{z'}(x)$ for all $x \in X$, then we treat z and z' as a single state z'' with utility $u(q, r; x, z'') = \pi(z)u(q, r; x, z) + \pi(z')u(q, r; x, z')$.) As noted above, our interpretation is that x is an idiosyncratic taste shock and z is a common taste shock. The realized type, x, plays two roles: it serves as private information about z and it is private information about preferences.

Finally, we make an additional assumption about the ex post limit version (where there is a continuum of agents and no uncertainty) of our economy. For each $z \in Z$, let $\mathcal{L}^z$ be a version with known aggregate state z and with fraction of type-x agents equal to $\mu_z(x)$. We assume that for each $z \in Z$, $\mathcal{L}^z$ has a regular competitive equilibrium (CE) in which every type trades. As explained further below, this assumption about trade, which is generic, is used to get differentiability of best responses in our market game.

Now, we turn to the mechanism. After types are realized, each agent n chooses an offer $a^n = (a^n_q, a^n_r) \in \mathcal{O}$, where

$$\mathcal{O} = \{(o_q, o_r) \in [0, \bar{q}] \times [0, \bar{r}] : o_q o_r = 0\}. \tag{1}$$

Then agents are randomly divided into two groups in the following way. Let $\eta \in (0, 1)$ and let $\lceil (1-\eta)N \rceil = M$ denote the smallest integer that is no less than $(1-\eta)N$. An assignment, which assigns a number n' to each agent $n \in \mathcal{N}$ in a one-to-one fashion, is drawn from the uniform distribution over the set of all such assignments, and agent n is called *active* if $n' \leq M$ and is called *inactive* if $n' > M$. (Notice that the identities of the inactive/active are random, but that M is a deterministic function of N.) The payoff for each inactive agent is given by trade at the fixed price, $p_1 = \bar{r}/\bar{q}$. That is,

$$(q^n, r^n) = (\bar{q} - a^n_q + a^n_r/p_1, \bar{r} - a^n_r + p_1 a^n_q) \text{ for } n \notin \mathcal{M}, \tag{2}$$

where $\mathcal{M}$ is the set of active agents. Next, the mechanism announces the histogram of the stage-1 offers of the active agents, denoted $\nu : \mathcal{O} \to \{0, 1, 2, ..., M\}$.[5] For each $a \in \mathcal{O}$, $\nu(a)$ is the number of active agents whose stage-1 offers are a. Then, given that information, the second stage has active agents participating in a market game. Each active agent n makes an offer $b^n = (b^n_q, b^n_r) \in \mathcal{O}$ and gets payoff

$$(q^n, r^n) = (\bar{q} - b^n_q + b^n_r/p_2, \bar{r} - b^n_r + p_2 b^n_q) \text{ for } n \in \mathcal{M}, \tag{3}$$

where $p_2 = R/Q$ and

$$(Q, R) = \sum_{n \in \mathcal{M}} b^n + M\kappa. \tag{4}$$

[4]This assumption also appears in most other related papers. Without this full support assumption, a signal x could perfectly reveal the state even without any information aggregation. If it did, the information structure would violate the "informational smallness" assumption in McLean and Postlewaite [8].

[5]We could let the mechanism announce two histograms, one for active agents and one for inactive agents. However, that would complicate the notation and would not change the results.

Here, $\kappa = (\kappa_q, \kappa_r) \in \mathbb{R}^2_{++}$ are exogenous (small) quantities that avoid the need to define payoffs when there are zero-offers on one side of the market, which also prevent no-trade from being an equilibrium, a formulation borrowed from Dubey and Shubik [3]. Notice that p_2 functions as a "price," but a price that depends both on the aggregation of offers from other active agents and on agent n's offer.[6] Moreover, because we require offers to be in the set $\mathcal{O}$, p_2 is not differentiable in agent n's offer at the offer $(0,0)$. That is why we assume that every type trades in at least one regular CE for the ex post limit economy, $\mathcal{L}^z$.

As it stands, this mechanism violates feasibility. The trades of inactive agents at the fixed price do not clear that market. In addition, resources are required for the positive κ_q and κ_r. We proceed as if the mechanism designer has the resources required to support the mechanism. In the concluding remarks, we suggest that a small entry fee could be used to provide those resources. In any case, the departure from feasibility (and *balancedness*) can be made arbitrarily small in per capita terms by choosing η and κ to be small.

The restriction in $\mathcal{O}$ that agents can only make offers on one side of the market plays a significant role in our analysis. It is used to obtain uniqueness of best responses. The following lemma shows that the restriction is not binding on the agent when there is no private information, which will be the case for the stage-2 game in the candidate equilibrium we construct.[7]

Lemma 1. Fix stage-2 offers of all other agents. Given those offers, for any offer $b' \in [0, \bar{q}] \times [0, \bar{r}]$, there exists $b'' \in \mathcal{O}$ that has the same payoff as b'.

Obviously, the restriction is also not binding in the same sense on payoffs for inactive agents.

The above mechanism is simple in the sense described in the introduction. First, as regards actions, the agents name quantities in at most two stages. Second, the mechanism makes use only of the agents' actions and assigns payoffs in the form of quantities using only addition and multiplication. Indeed, the main difference between our mechanism and pari-mutuel betting is the exogenous and random selection of some agents who do not get to participate in the second stage.

3 Separating equilibrium: existence

We begin our analysis with a definition of strategies and equilibrium. A stage-1 strategy is $s_1^n(x) \in \mathcal{O}$, while a stage-2 strategy is $s_2^n(x, a, \nu^{-a}) \in \mathcal{O}$, where the second component in the domain is the agent's stage-1 action, and the third is the announced histogram of offers of active agents *net of the agent's own action.* (That is, for any $a' \in \mathcal{O}$, $\nu^{-a}(a') = \nu(a')$ if $a \neq a'$ and $\nu^{-a}(a) = \nu(a) - 1$.) A strategy profile $\{(s_1^n, s_2^n) : n \in \mathcal{N}\}$ is a perfect bayesian equilibrium (PBE) if for each $n \in \mathcal{N}$, s_1^n is a best response to $\{(s_1^{n'}, s_2^{n'}) : n' \neq n\}$ and

[6] Our market game is a version of what is known as the "buy-sell" game, as opposed to the "sell-all" version (see Shapley-Shubik [12]).

[7] All proofs appear in the Appendix.

s_2^n is a best response to $\{s_2^{n'} : n' \neq n\}$ with respect to a belief φ^n that is consistent with Bayes' rule whenever possible.

Throughout, we focus on symmetric equilibrium in pure strategies. A PBE is a *symmetric equilibrium* if for all $n \in \mathcal{N}$, $(s_1^n, s_2^n) = (s_1, s_2)$ and $\varphi^n = \varphi$. In a symmetric equilibrium, an agent's expected payoff at stage-2 depends only on his private history (x, a) and the configuration of other active agents' private histories $\theta : X \times \mathcal{O} \to \{0, 1, 2, ..., M-1\}$. Thus, we may formulate the belief $\varphi(x, a, \nu^{-a})$ as an element of $\Delta(Z \times \Theta)$, where Θ is the set of all configurations θ of type/stage-1-action of the other active agents. Then, a symmetric equilibrium is a triple (s_1, s_2, φ) such that (a) $s_1(x)$ is a best response to s_1 and s_2; (b) $s_2(x, a, \nu^{-a})$ is a best response to s_2 and $\varphi(x, a, \nu^{-a})$; (c) $\varphi(x, a, \nu^{-a})$ is derived from equilibrium behavior using Bayes' rule whenever possible.

A *separating* equilibrium is a symmetric equilibrium in which $x \neq y$ implies $s_1(x) \neq s_1(y)$. In a separating equilibrium, all active agents share the same belief on the equilibrium path. In particular, in such an equilibrium, ν^{-a} and the agent's own type reveal the true configuration of types of the active agents and each agent uses that true configuration and Bayes' rule to form a common posterior over Z. Thus, on the equilibrium path in a separating equilibrium, although the agent's private type matters for the agent's preferences at stage 2, all active agents share the same information at that stage.

We show that a separating equilibrium exists generically for sufficiently large N. We establish existence by demonstrating that it is optimal for an agent at the first stage to choose an action that is best contingent on being inactive when others do so. The genericity qualification is simple: as spelled out below, it says that conditional on being inactive, different types make different stage-1 offers. We begin with existence and characterization of the stage-2 equilibrium when stage-1 is separating. Those results along with explicit stage-1 strategies and beliefs are used to construct the candidate equilibrium.

3.1 Stage-2 when stage-1 is separating

In a separating equilibrium, the belief about the type/stage-1-action configuration of other active agents is degenerate on the configuration θ given by $\theta(x, s_1(x)) = \nu^{-a}(s_1(x))$ on the equilibrium path. This implies that there is common knowledge at stage 2 about the type-configuration of active agents, a configuration we denote $\sigma : X \to \{0, 1, ..., M\}$, where M is the number of active agents.[8] It also implies a common posterior over Z, denoted ϕ^σ, which is derived from the type-configuration σ via Bayes' rule. Therefore, the stage-2 game in a separating equilibrium depends only on the known type-configuration σ.

Another implication, which will be useful for our efficiency results, is that a separating equilibrium achieves complete information about the state-of-the-world asymptotically. Fix $z \in Z$ and let σ^N be the type configuration of active agents for an economy of size N. If the sequence $\{\sigma^N\}_{N=1}^{\infty}$ is such that $\lim_{N\to\infty} \sigma^N(x)/\lceil(1-\eta)N\rceil = \mu_z(x)$ for each $x \in X$ (which holds *almost surely* conditional on z), then $\lim_{N\to\infty} \phi^{\sigma^N}(z) = 1$. This follows from

[8] In what follows, and only to simplify notation, we assume that $\sigma(x) > 0$ for all $x \in X$. Obviously, this holds with arbitrarily high probability for sufficiently large N.

our informativeness assumption—namely, that for any $z \neq z'$, there exists some x such that $\mu_z(x) \neq \mu_{z'}(x)$—and the full-support assumption that $\mu_z(x) > 0$ for all x, z.

Because we only focus on symmetric equilibrium, the stage-2 game is a one-shot game that depends only on the type-configuration σ. However, for our purposes, it is convenient to treat it as a game that depends on four parameters, $(M, \mu, \phi, \kappa) \in \mathbb{N} \times \Delta(X) \times \Delta(Z) \times \mathbb{R}^2_{++}$, where the first three are determined by σ—a game denoted $\mathcal{E}(M, \mu, \phi, \kappa)$.

The game $\mathcal{E}(M, \mu, \phi, \kappa)$. There are M players; the action set for each player is $\mathcal{O}$; the known number of players of type x is $M\mu(x)$; there are per capita exogenous offers $\kappa = (\kappa_q, \kappa_r)$; the payoff for type-$x$ players is the expected value of u w.r.t. the common prior $\phi \in \Delta(Z)$ evaluated at the bundle implied by the stage-2 market game (see (3)).

We have two results regarding the game $\mathcal{E}(M, \mu, \phi, \kappa)$. The first establishes existence of a symmetric equilibrium. The second, which uses the dependence on the parameters, establishes a lower hemicontinuity result w.r.t. a limit economy. We use both to establish existence of a separating equilibrium: the first gives a stage-2 equilibrium for an arbitrary type-configuration, while the second allows us to construct coordinated stage-2 behavior for type-configurations that are close to the limit distributions of (σ/M).

Here, then, is the general existence result.

Proposition 1. Let $\mathcal{B}$ denote the set of symmetric equilibria for the game $\mathcal{E}(M, \sigma/M, \phi, \kappa)$. If $\kappa > 0$, then $\mathcal{B}$ is not empty.

The proof is a routine application of Brouwer's fixed point theorem. It does, however, depend crucially on the constraint $b_q b_r = 0$. With it, the best response, which is the mapping studied in order to get a fixed point, is a function; without that constraint, the mapping is not necessarily a convex correspondence.[9] As noted above, $\kappa > 0$ allows us to avoid the well-known difficulties that would otherwise arise when there are zero offers by others.

In addition to Proposition 1, in order to prove existence of a separating equilibrium, we also make use of a characterization of the equilibria in $\mathcal{E}(M, \sigma/M, \phi^\sigma, \kappa)$ for M large and σ/M close to μ_z. For small κ, our next result provides that characterization by establishing a lower-hemicontinuity result for the Proposition 1 equilibrium-correspondence at $(M, \mu, \phi, \kappa) = (\infty, \mu_z, \delta_z, \kappa)$, where δ_z is the dirac measure concentrated at z. We include μ and ϕ in the domain of the correspondence because we do not have deterministic replication of the number of types. With (M, μ, ϕ) as the domain, such lower-hemicontinuity and the fact that $\lim_{M\to\infty}(\sigma/M, \phi^\sigma) = (\mu_z, \delta_z)$ almost surely conditional on z suffice for our existence result.

The economy at $(M, \mu, \phi, \kappa) = (\infty, \mu_z, \delta_z, \kappa)$ is denoted $\mathcal{L}^z(\kappa)$. It is an economy with z known, with fraction of type-x agents equal to $\mu_z(x)$ for all $x \in X$, and with exogenous per capita trades κ. We describe candidate equilibria for $\mathcal{L}^z(\kappa)$ in terms of competitive equilibria for $\mathcal{L}^z(\kappa)$. As is standard, a CE for $\mathcal{L}^z(\kappa)$ is denoted $\langle p^{z,\kappa}, (q_x^{z,\kappa}, r_x^{z,\kappa})_{x\in X}\rangle$ and

[9]Dubey and Shubik [3] give a similar existence result. However, instead of assuming $b_q b_r = 0$ as a constraint on agents, they use Lemma 1 to select the trade that satisfies that constraint from the set of all maximizing trades.

satisfies

$$\frac{\kappa_r}{p^{z,\kappa}} + \sum_{x \in X} \mu_z(x) q_x^{z,\kappa} = \bar{q} + \kappa_q,$$

and is such that $(q_x^{z,\kappa}, r_x^{z,\kappa})$ maximizes $u(q, r; x, z)$ subject to $p^{z,\kappa} q + r = p^{z,\kappa} \bar{q} + \bar{r}$. Then, the corresponding candidate stage-two equilibrium is given by the offers $\beta^{z,\kappa} = \langle \beta_q^{z,\kappa}(x), \beta_r^{z,\kappa}(x) \rangle_{x \in X}$, where

$$\beta_q^{z,\kappa}(x) = \max\{\bar{q} - q_x^{z,\kappa}, 0\} \text{ and } \beta_r^{z,\kappa}(x) = \max\{\bar{r} - r_x^{z,\kappa}, 0\}. \tag{5}$$

Finally, because ∞ is not well defined, we replace (M, μ, ϕ, κ) by $(1/M, \mu, \phi, \kappa)$ and allow the first parameter to vary over $[0, 1]$. Then we have the following lower-hemicontinuity result.

Proposition 2. Fix $z \in Z$ and fix a regular CE for $\mathcal{L}^z$ where every type trades, $\langle p^{z,0}, (q_x^{z,0}, r_x^{z,0})_{x \in X} \rangle$. Let $A = [0, 1] \times \Delta(X) \times \Delta(Z)$ and let $a_0^z = (0, \mu_z, \delta_z)$, where δ_z is the dirac measure concentrated at z. There exists $\bar{\kappa} = (\bar{\kappa}_q, \bar{\kappa}_r) > 0$ (not dependent on z) such that if $\kappa \in (0, \bar{\kappa}]$, then

(i) the economy $\mathcal{L}^z(\kappa)$ has a regular CE, $\langle p^{z,\kappa}, (q_x^{z,\kappa}, r_x^{z,\kappa})_{x \in X} \rangle$, in which every type trades, and whose corresponding offer, denoted $\beta^{z,\kappa}$ (see (5)), is continuous in κ (even at $\kappa = (0, 0)$);

(ii) there exists an open neighborhood of a_0^z, denoted $A_{z,\kappa} \subset A$, and a continuous function $f_{z,\kappa} : A_{z,\kappa} \to \mathcal{O}^X$ such that $f_{z,\kappa}(a_0^z) = \beta^{z,\kappa}$ and $(1/M, \sigma/M, \phi^\sigma) \in A_{z,\kappa}$ implies that $f_{z,\kappa}(1/M, \sigma/M, \phi^\sigma)$ is a symmetric equilibrium in $\mathcal{E}(M, \sigma/M, \phi^\sigma, \kappa)$.

The proof is an application of the implicit function theorem.[10] Part (i), which we do not prove, is the well-known result that existence of a regular CE in which each type trades in $\mathcal{L}^z$, one of our assumptions, implies existence of a regular CE in which each type trades in $\mathcal{L}^z(\kappa)$ for all sufficiently small κ. Positive trade implies differentiability of best responses in the stage-2 game, $\mathcal{E}(M, \sigma/M, \phi^\sigma, \kappa)$, in a neighborhood of a_0^z. That, in turn, permits us to prove part (ii) by invoking the implicit function theorem at a_0^z.[11]

3.2 Existence of separating equilibrium

Contingent on being inactive, an agent of type-x at stage-1 chooses $a \in \mathcal{O}$ to maximize

$$G_x(a) = \sum_{z \in Z} \tau_x(z) u(\bar{q} - a_q + a_r/p_1, \bar{r} - a_r + p_1 a_q; x, z). \tag{6}$$

By the argument in the proof of Proposition 1, a unique maximum of $G_x(a)$ exists. We denote it $\alpha^* = \{\alpha^*(x)\}_{x \in X}$. Generically, α^* is separating in the sense that $x \neq y$ implies

[10] If we drop the every-type-trades assumption, then this result may go through if we allow agents to use any offer $(o_q, o_r) \in [0, \bar{q}] \times [0, \bar{r}]$. While this would avoid issues with differentiability, it would require us to use a version of the implicit function theorem that applies to correspondences.

[11] The only other lower hemicontinuity result for market games seems to be that in Mas-Colell [9], but he uses a very different model. As he says, his version is not a game because each agent's offers are constrained by ex post budget balance, a constraint which depends on the actions of others.

$\alpha^*(x) \neq \alpha^*(y)$. Indeed, if α^* is not separating, then for some $x \neq y$, $\alpha^*(x) = a^* = \alpha^*(y)$, and

$$\frac{\sum_{z\in Z} \tau_x(z) u_q(q^*, r^*; x, z)}{\sum_{z\in Z} \tau_x(z) u_r(q^*, r^*; x, z)} = p_1 = \frac{\sum_{z\in Z} \tau_y(z) u_q(q^*, r^*; y, z)}{\sum_{z\in Z} \tau_y(z) u_r(q^*, r^*; y, z)},$$

where $q^* = \bar{q} - a_q^* + \frac{a_r^*}{p_1}$ and $r^* = \bar{r} - a_r^* + p_1 a_q^*$. But this restriction holds only for knife-edge cases for two distinct aspects of the environment: the probabilities, $\tau_x(\cdot)$ and $\tau_y(\cdot)$, and the utilities, $u(q^*, r^*; x, \cdot)$ and $u(q^*, r^*; y, \cdot)$.

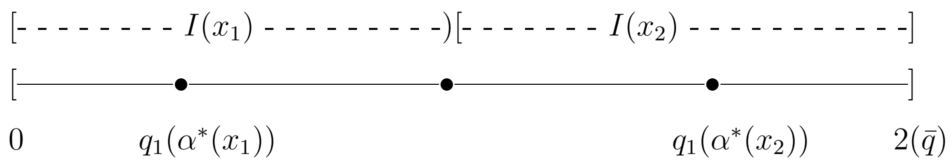
Now we describe candidate beliefs under the assumption that α^* is separating. For any $a \in \mathcal{O}$, let

$$q_1(a) = \bar{q} - a_q + \frac{a_r}{p_1} \in [0, 2\bar{q}]. \tag{7}$$

By separation of α^*, $x \neq y$ implies $q_1(\alpha^*(x)) \neq q_1(\alpha^*(y))$. Therefore, we can order the elements of X so that $q_1(\alpha^*(x_i)) < q_1(\alpha^*(x_{i+1}))$ for $i \in \{1, 2, ..., |X| - 1\}$, where $|X|$ denotes the cardinality of X. Next, we partition the interval $[0, 2\bar{q}]$ into $|X|$ subintervals indexed by that ordering as follows:

$$I(x_i) = \begin{cases} \left[0, \frac{q_1(\alpha^*(x_2)) + q_1(\alpha^*(x_1))}{2}\right) \text{ for } i = 1 \\ \left[\frac{q_1(\alpha^*(x_i)) + q_1(\alpha^*(x_{i-1}))}{2}, \frac{q_1(\alpha^*(x_{i+1})) + q_1(\alpha^*(x_i))}{2}\right) \text{ for } i = 2, 3, ..., |X| - 1 \\ \left[\frac{q_1(\alpha^*(x_i)) + q_1(\alpha^*(x_{i-1}))}{2}, 2\bar{q}\right] \text{ for } i = |X| \end{cases}. \tag{8}$$

For $|X| = 2$, $I(x_1)$ and $I(x_2)$ are depicted as follows:



An agent's belief is a joint distribution over the type/stage-1-action configuration of the other active agents and the state-of-the-world z. It is derived from the observed histogram, ν, and from knowledge of the agent's own private information and is defined for arbitrary stage-1 outcomes. Our candidate for equilibrium beliefs is that each agent forms a degenerate distribution over the type/stage-1-action configuration of the other active agents by treating an observed stage-1 action in $I(x_i)$ as coming from an agent of type x_i. Here is the formal description of the beliefs.

Candidate for equilibrium beliefs. The belief for an agent with private history (x, a) and observing histogram ν, denoted $\varphi^*(x, a, \nu^{-a})$, puts probability 1 on the configuration $\theta_{\nu^{-a}}$ defined by

$$\theta_{\nu^{-a}}(y, a') = \begin{cases} \nu^{-a}(a') \text{ if } q_1(a') \in I(y) \\ 0 \text{ otherwise} \end{cases}. \tag{9}$$

An associated marginal distribution over Z is given by the posterior derived from Bayes' rule using the type-configuration of all active agents $\sigma^* : X \to \{0, 1, ..., M\}$ defined by

$$\sigma^*(y) = \begin{cases} \sum_{a' \in \mathcal{O}} \theta_{\nu^{-a}}(y, a') \text{ if } y \neq x \\ \sum_{a' \in \mathcal{O}} \theta_{\nu^{-a}}(x, a') + 1 \text{ if } y = x \end{cases}. \tag{10}$$

Next, we construct candidate equilibrium strategies. This involves two steps. First, we use Propositions 1 and 2 to map each realized type-configuration into a stage-2 equilibrium. Then we construct the strategies.

Candidate for stage-2 equilibrium. Fix $\kappa \in (0, \bar{\kappa}]$, where $\bar{\kappa}$ satisfies Proposition 2. Recall that, by Proposition 1, for any given type configuration σ and any M, the set of symmetric equilibrium in the game $\mathcal{E}(M, \sigma/M, \phi^\sigma, \kappa)$, denoted $\mathcal{B}$, is not empty. Moreover, Proposition 2 (i) guarantees that, for each $z \in Z$, $\mathcal{L}^z(\kappa)$ has a regular CE, $\langle p^{z,\kappa}, (q_x^{z,\kappa}, r_x^{z,\kappa})_{x \in X} \rangle$, where every type trades, and Proposition 2 (ii) guarantees that for M sufficiently large and for σ's (for M active agents) close to the limit distribution μ_z, a symmetric equilibrium exists close to the CE. Given these results, we let

$$\beta^\sigma = \begin{cases} f_{z,\kappa}(1/M, \sigma/M, \phi^\sigma) \text{ if } (1/M, \sigma/M, \phi^\sigma) \in A_{z,\kappa} \text{ for some } z \in Z \\ \text{an arbitrary element of } \mathcal{B} \text{ otherwise} \end{cases}. \tag{11}$$

Notice that this is well-defined iff $A_{z,\kappa} \cap A_{z',\kappa} = \emptyset$ for all $z \neq z'$. Because $a_0^z \neq a_0^{z'}$ for all $z \neq z'$, we can choose the $A_{z,\kappa}$'s to be disjoint.

We need to specify an equilibrium stage-2 offer for any realization of σ because types are random. Proposition 1 provides existence of a symmetric stage-2 equilibrium for any σ. However, to construct a separating equilibrium, we use Proposition 2 to find a symmetric stage-2 equilibrium for σ's close to limit configurations. By taking σ as a parameter, Proposition 2 (ii) shows that for any $z \in Z$ and *any* sequence $\{\sigma^M\}_{M=1}^\infty$ such that $\lim_{M\to\infty} \sigma^M/M = \mu_z$, we have $\lim_{M\to\infty} \beta^{\sigma^M} = \beta^{z,\kappa}$, where $\beta^{z,\kappa}$ is the offer corresponding to the CE, $\langle p^{z,\kappa}, (q_x^{z,\kappa}, r_x^{z,\kappa})_{x \in X} \rangle$, in $\mathcal{L}^z(\kappa)$ given by Proposition 2 (i).

We describe the candidate for equilibrium strategies in terms of the β^σ mapping given by (11). To do that, it is helpful to distinguish between two classes of active agents according to their private histories. We call a type-x agent a *nondefector* if the agent's stage-1 action is in $I(x)$; otherwise, the agent is called a *defector*. Notice that if no one defects, then all agents' beliefs are symmetric in the sense assumed in Propositions 1 and 2: all have the same posterior on z and all active agents have the same belief about the type-configuration over all active agents, which happens to be the true configuration. If one agent defects or more than one defect, then all nondefectors have symmetric beliefs; they have the same posterior on z and the same belief about the type-configuration over all active agents—which, however, is not the true configuration. Each defector has a different posterior on z and a different belief about the type-configuration for the active agents.

The belief φ^* has each agent believing that other agents do not defect. Our specification for a candidate equilibrium is consistent with that belief. In particular, our candidate stage-2 strategy, which must be defined for arbitrary stage-1 actions, has each agent believing that other agents did not defect at stage-1.

Candidate for equilibrium strategies. For stage-1, our candidate is

$$s_1^*(x) = \alpha^*(x), \tag{12}$$

the offer that maximizes $G_x(a)$ (see (6)). For stage-2 strategies, consider an agent with private history (x, a, ν^{-a}) and $q_1(a) \in I(x')$. Let σ^* be the agent's belief about the type-configuration of all active agents under φ^* and let σ' be the type-configuration that he believes other agents believe (see (10)). (If $x' = x$ (nondefector), then $\sigma'(y) = \sigma^*(y)$ for all y; otherwise (defector), $\sigma'(x) = \sigma^*(x) - 1$, $\sigma'(x') = \sigma^*(x') + 1$, and $\sigma'(y) = \sigma^*(y)$ for all $y \notin \{x, x'\}$.) Then, $s_2^*(x, a, \nu^{-a})$ satisfies

$$s_2^*(x, a, \nu^{-a}) \in \arg\max_{b \in \mathcal{O}} \sum_{z \in Z} \phi^{\sigma^*}(z) u(\bar{q} + \frac{b_r Q_- - b_q R_-}{R_- + b_r}, \bar{r} + \frac{b_q R_- - b_r Q_-}{Q_- + b_q}; x, z), \tag{13}$$

where $\phi^{\sigma^*}(z)$ is derived from σ^* using Bayes' rule and where

$$(Q_-, R_-) = M\kappa + \sum_{y \neq x} \sigma^*(y) \beta^{\sigma'}(y) + (\sigma^*(x) - 1)\beta^{\sigma'}(x).$$

Notice that if the agent is a nondefector so that $\sigma' = \sigma^*$, then $s_2^*(x, a, \nu^{-a}) = \beta^{\sigma^*}(x)$ as given by (11) for $\sigma = \sigma^*$.

Theorem 1. Suppose that $\alpha^*(x) \neq \alpha^*(y)$ for any $x \neq y$ and that $\kappa \in (0, \bar{\kappa}]$, where $\bar{\kappa}$ satisfies Proposition 2. There exists $\bar{N}$ (which depends on κ) such that if $N \geq \bar{N}$, then the N-agent economy has a separating equilibrium.

The proof shows that the above candidate is an equilibrium. By construction, $s_1^* = \alpha^*$ implies that φ^* is consistent with Bayes' rule. Also, by construction, $s_2^*(x, a, \nu^{-a})$ is a best response to s_2^* with respect to φ^*. That follows because, according to φ^*, the agent believes that every other active agent is a nondefector, following any stage-1 actions. And, if they follow s_2^*, then their actions are described by $\beta^{\sigma'}$. Therefore, what remains, and is the focus of the proof, is to show that α^* is optimal given that other agents follow the candidate equilibrium. An agent at the first stage faces a tradeoff. Conditional on being inactive, playing α^* is optimal for any N. Conditional on being active, a type-x agent could gain by playing something not in $I(x)$. By doing that, the agent influences the beliefs and, thereby, the stage-2 actions of other active agents. To demonstrate that the trade-off is resolved in favor of the inactive case, we first show that an agent's expected stage-2 payoff conditional on each state z converges to the payoff according to the CE, $\langle p^{z,\kappa}, (q_x^{z,\kappa}, r_x^{z,\kappa})_{x \in X} \rangle$ (see the first line of (11)) independent of his offer made at stage-1. In this argument only those σ's that are close to the limit configurations are relevant as they occur with probability close to one as the population gets large. Hence, the arbitrariness of the selection in the second line of (11) does not matter. As a result, the gain from manipulating stage-2 beliefs is smaller than the loss implied by playing something that is not in $I(x)$—a play which, by construction, is bounded away from $\alpha^*(x)$.

The threat of being inactive plays a crucial role in the proof. Without it, there would be no penalty attached to stage-1 actions that are devoted entirely to manipulating the beliefs of others and such manipulation could be desirable for any finite N. Therefore, we strongly suspect that a separating equilibrium does not exist if $\eta = 0$. In this respect, there is a significant distinction between the model with a finite number of agents and the same model with a continuum of agents. In the continuum version as usually formulated, one agent cannot manipulate the beliefs of others and a separating equilibrium exists even if $\eta = 0$.

Finally, as is apparent from the proofs of Proposition 2 and Theorem 1, our arguments do not rely on a particular choice of the CE for each state z. Thus, for any selection of a regular CE for $\{\mathcal{L}^z\}_{z\in Z}$ in which every type trades, we can construct a corresponding separating equilibrium in our mechanism for N sufficiently large. Moreover, by Proposition 2, the stage-2 outcome in such a corresponding equilibrium converges almost surely to the designated CE as N goes to infinity for each state z. The following corollary summarizes these results.

Corollary 1. Suppose that $\alpha^*(x) \neq \alpha^*(y)$ for any $x \neq y$. For each $z \in Z$, let $\langle p^{z,0}, (q_x^{z,0}, r_x^{z,0})_{x\in X}\rangle$ be a regular CE in which every type trades in $\mathcal{L}^z(0)$. In addition, fix $\kappa \in (0, \bar{\kappa}]$, where $\bar{\kappa}$ is given in Proposition 2, and let $(q_x^{z,\kappa}, r_x^{z,\kappa})_{x\in X}$ be the associated (nearby) CE allocation in $\mathcal{L}^z(\kappa)$. There exists $\bar{N}$ and a sequence of separating equilibria $\{(s_1^N, s_2^N)\}_{N=\bar{N}}^{\infty}$, indexed by the number of agents N, for which the following holds. Let $(q_x^N, r_x^N)_{x\in X}$ be the stage-2 consumption for agents of type-x in (s_1^N, s_2^N). Then, for each $z \in Z$,

$$\lim_{N\to\infty} (q_x^N, r_x^N)_{x\in X} = (q_x^{z,\kappa}, r_x^{z,\kappa})_{x\in X}$$

almost surely conditional on z.

4 Uniqueness of equilibrium

As shown in Corollary 1, if there are multiple regular competitive equilibria in the ex post limit economy, $\mathcal{L}^z$, then there are also multiple separating equilibria in our mechanism. Here we show that if competitive demand for good-q is monotone in $\mathcal{L}^z$ for each $z \in Z$,[12] and if some mild additional conditions hold, then any equilibrium is separating and can be characterized asymptotically by the unique CE. There are three such conditions.

The first is a stronger assumption about the informativeness of signals.

A1. Let $\mathcal{Y} = \{Y_1, Y_2\}$ be any bipartition of X and let $\mu_z(Y_i) \equiv \sum_{y\in Y_i} \mu_z(y)$. For any $z \neq z'$, $\mu_z(Y_1) \neq \mu_{z'}(Y_1)$.

This implies that for any partition $\mathcal{Y} = \{Y_1, ..., Y_K\}$ of X with $K \geq 2$ and for any $z \neq z'$, there exists some k such that $\mu_z(Y_k) \neq \mu_{z'}(Y_k)$. Although this assumption is stronger than our original informativeness assumption, parameters for which it does not hold are nongeneric.

The second is a modification of the mechanism.

[12]This monotonicity assumption helps simplify our notation and the statement of results, but is not essential for our uniqueness result. As noted below, uniqueness of CE in $\mathcal{L}^z$ would suffice.

A2. If all agents make the same stage-1 offer, then all active agents are required to make zero offers at the second stage.

This says that the market shuts down after the first stage if all agents announce the same offer at the first stage. Under the assumption that $\alpha^*(x) \neq \alpha^*(y)$ for all $x \neq y$, this modification rules out any equilibrium (s_1, s_2) such that $s_1(x) = s_1(y)$ for all $x, y \in X$, but does not change any other symmetric equilibrium if it exists. In particular, this modification does not affect the existence of a separating equilibrium. Moreover, our mechanism is still robust to the details of the environment under this modification. Finally, because all agents have the same endowments, in the rare event that every agent receives the same signal in a separating equilibrium, such shutting down is costless in terms of realized welfare because in that rare event there is no role for trade.

The third is a restriction on off-equilibrium beliefs.

A3. If a single deviating offer is observed at the first stage, then it is believed to come from some set of types $A \subset X$. Moreover, that belief and the equilibrium play of other agents are used via Bayes' rule to form a belief over Z and the type configuration of other active agents.

Along the equilibrium path of a symmetric equilibrium in pure strategies, the equilibrium belief associates each equilibrium stage-1 offer a with a set of types and then applies Bayes' rule to derive a belief about the type configuration and the state. A3 requires off-equilibrium beliefs to be derived using the same procedure, but allows there to be an arbitrary set of types, A, to be to be associated with an arbitrary deviating offer. The assumption that A is common to all nondeviators is convenient, but not crucial. The crucial part of A3 is that a set of types is assumed for the deviator and that Bayes' rule is used based on that set. As a result, A3 excludes off-equilibrium beliefs that allow the deviator to signal something about other agents' types or about the state in a way that is not warranted by the deviator's private information. This requirement is essentially the requirement for "reasonable" belief systems in Fudenberg and Tirole [4]. Their requirement says that inferences drawn from a deviating action should be limited to the deviator's type (that is, no signaling about what you don't know). We need to augment their requirement with the use of Bayes' rule because of the presence in our model of a payoff-relevant state-of-the-world. Doing so is reasonable because an agent is trying to update his belief about the types and the state-of-the-world, which are exogenous.[13]

Theorem 2. Suppose that $\alpha^*(x) \neq \alpha^*(y)$ for any $x \neq y$, that the competitive demand for good-q is monotone in $\mathcal{L}^z$ for each $z \in Z$, and that A1-A3 hold. Fix $\kappa > 0$. There exists $\bar{N}$ such that if $N > \bar{N}$, then any equilibrium $s^N = (s_1^N, s_2^N)$ is separating.[14]

[13] A less restrictive extension would allow the off-equilibrium belief to associate a deviating offer with a *distribution* of types and then employ Bayes' rule to pin down the belief about the state and the type configuration of other agents. However, it is rather complicated to formulate the use of Bayes' rule under this assumption and doing so does not seem to affect our main results.

[14] The monotonicity of the demand functions guarantees that we have a unique CE in $\mathcal{L}^z(\kappa)$ for any κ. Therefore, in Theorem 2, we do not need κ to be small. However, small k would be needed had we only assumed that $\mathcal{L}^z$ has a unique CE which is also regular (but need not have every type trade). Then, we could only guarantee that $\mathcal{L}^z(\kappa)$ has a unique CE for κ sufficiently small.

We show by contradiction that any equilibrium is separating for sufficiently large N. First, we use A2 to eliminate a complete pooling equilibrium—one in which $s_1(x) = s_1(y)$ for all $x, y \in X$. Next, we consider a semi-pooling equilibrium—one in which there is a partition $\mathcal{Y} = \{Y_1, ..., Y_K\}$ of X with $|X| > K \geq 2$ such that $s_1(y) = s_1(y')$ if $y, y' \in Y_k$ and $s_1(y) \neq s_1(y')$ if $y \in Y_k$ and $y' \in Y_{k'}$ with $k \neq k'$. The main body of the proof shows that, even under a semi-pooling equilibrium, the stage-2 outcome converges to the unique competitive allocation. Of course, this convergence will fail without uniqueness of the CE. We also make use of A1 and A3: A1 is used to deal with the asymmetric information that exists in a semi-pooling equilibrium, while A3 is used to restrict off-equilibrium beliefs. Then we eliminate any semi-pooling equilibrium by an argument that resembles the main idea of the proof of Theorem 1: because a deviation by one agent has a vanishing effect on the beliefs of other agents, an agent is induced to defect from a semi-pooling equilibrium and play the stage-1 strategy that is best contingent on becoming inactive.

Theorem 2 shows that when the population is sufficiently large, only separating equilibrium can occur in our mechanism. Moreover, for sufficiently large populations, in any equilibrium the stage-1 behavior is characterized by α^* and the stage-2 behavior is *almost surely* close to "price-taking" with respect to a price that is close to the unique CE price. The following corollary summarizes these observations.

Corollary 2. Suppose that the conditions in Theorem 2 hold. For any sequence of equilibria $\{(s_1^N, s_2^N)\}_{N=1}^{\infty}$, if $(q_x^N, r_x^N)_{x \in X}$ is the stage-2 consumption for agents of type-x in (s_1^N, s_2^N), then, for each $z \in Z$,

$$\lim_{N \to \infty} (q_x^N, r_x^N)_{x \in X} = (q_x^{z,\kappa}, r_x^{z,\kappa})_{x \in X}$$

almost surely conditional on z, where $(q_x^{z,\kappa}, r_x^{z,\kappa})_{x \in X}$ is the competitive allocation in $\mathcal{L}^z(\kappa)$.

Corollaries 1 and 2 may be regarded as "strategic foundations" for rational-expectations equilibrium. They show that our mechanism implements the ex post competitive equilibrium allocations asymptotically for the active agents. When there are many such allocations, Corollary 1 shows that any of them is an equilibrium of our mechanism. When there is a unique such allocation, our mechanism has an essentially unique equilibrium giving that allocation. In the next section, we show that when both κ and η are small, our mechanism is almost ex post efficient taking active and inactive agents into account.

5 Almost ex post optimality

In our model, there are three sources of exogenous uncertainty, described by three random variables: the profile of agents' types, denoted $\zeta^N = (\zeta_1, ..., \zeta_n, ...\zeta_N) \in X^N$; the profile of agents' activeness status, denoted $c^N = (c_1, ..., c_N)$, where $c^N \in \mathbb{C}^N = \{c^N \in \{0,1\}^N : \sum_{n \in \mathcal{N}} c_n = \lceil (1-\eta)N \rceil\}$, and $c_n = 0$ means inactive and $c_n = 1$ means active; and the state of the world, $z \in Z$. An allocation, denoted $\omega = \langle \omega_n = (q_n, r_n) : n \in \mathcal{N} \rangle$, is a mapping from these three random variables to consumption bundles for all agents. We define almost ex post efficiency by first defining notions of feasibility and ex post pareto-superiority.

Definition. Let $\omega : X^N \times \mathbb{C}^N \times Z \to (\mathbb{R}^2_+)^N$ be an allocation.

(i) We say that ω is $\delta = (\delta_q, \delta_r)$-*feasible* if

$$\sum_n \omega_n(\zeta^N, c^N, z) \leq N(\bar{q} + \delta_q, \bar{r} + \delta_r) \tag{14}$$

for all $(\zeta^N, c^N, z) \in X^N \times \mathbb{C}^N \times Z$.

(ii) Let $\{Y(z, c^N)\}$ be a collection of subsets of X^N, one subset for each $(z, c^N) \in Z \times \mathbb{C}^N$. We say that ω' is *ex post* ε*-pareto-superior to* ω w.r.t. the collection $\{Y(z, c^N)\}$ if

$$u[\omega'_n(\zeta^N, c^N, z); \zeta_n, z] > u[\omega_n(\zeta^N, c^N, z); \zeta_n, z] + \varepsilon \text{ for each } n \tag{15}$$

for some (z, c^N) and some $\zeta^N \in Y(z, c^N)$.

(iii) We say that a δ-feasible ω is *ex post* (ε, δ)*-efficient* if there exists a collection $\{Y(z, c^N)\}$ such that $\mathbb{P}[Y(z, c^N)|z, c^N] \geq 1 - \varepsilon$ for each (z, c^N) and such that no other δ-feasible allocation ω' is ex post ε-pareto-superior to ω w.r.t. $\{Y(z, c^N)\}$.

When $\varepsilon = \delta = 0$, the above definition coincides with the usual definition of ex post efficiency.[15] And, except for the presence of c^N, if $\delta = 0$, then this definition coincides with the definitions in McLean and Postlewaite [8] and in Gul and Postlewaite [6]. We use $\delta > 0$ to reflect the resources that the mechanism designer may need to run the mechanism. (In the concluding remarks, we discuss how to amend the mechanism so that it achieves almost ex post efficiency with $\delta = 0$.)

Here, then, are our efficiency results.

Theorem 3. Suppose that $\alpha^*(x) \neq \alpha^*(y)$ for any $x \neq y$. Let $\varepsilon > 0$ be given.

(i) Suppose that for each $z \in Z$, $\mathcal{L}^z$ has a regular competitive equilibrium where every type trades. There exists $\bar{\kappa}$ and $N(\kappa, \eta)$ such that if (κ, η) satisfies

$$\bar{\kappa} > \kappa \geq 4\eta(\bar{q}, \bar{r}) > 0 \tag{16}$$

and $N > N(\kappa, \eta)$, then there exists a separating equilibrium whose outcome is ex post $(\varepsilon, 1.5\kappa)$-efficient.

(ii) Suppose that the competitive demand for good-q is monotone in $\mathcal{L}^z$ for all $z \in Z$, and that A1-A3 hold. There exists $\bar{\kappa}$ and $N(\kappa, \eta)$ such that if (κ, η) satisfies (16) and $N > N(\kappa, \eta)$, then the outcome of any symmetric equilibrium in pure strategies is ex post $(\varepsilon, 1.5\kappa)$-efficient.

We sketch the proof of part (ii). (The proof for part (i) is similar, but must take into account that there may be more than one CE for $\mathcal{L}^z$ as defined in section 3.1.) We construct the collection $\{Y(z, c^N)\}$ for each realization (z, c^N) by choosing the type configurations that are close to the limit distribution conditional on z and for which the

[15] Notice that when $\varepsilon = 0$, (15) requires the allocation $\omega'_n(\zeta^N, c^N, z)$ to be strictly better than $\omega_n(\zeta^N, c^N, z)$ for all $n \in \mathcal{N}$. This is without loss of generality: any allocation that is weakly better off for all n and strictly better for some n can be modified to be strictly better off for all n because goods are divisible and utilities are continuous.

equilibrium stage-2 offers are close to the CE in the limit economy. Such an event has arbitrarily high probability because of the convergence results in Theorem 2 and Corollary 2. After the construction, we use the first fundamental welfare theorem to prove the efficiency result in the following way.

Consider a type-configuration (including both active and inactive agents) realization, σ^N, belonging to the event $Y(z, c^N)$, and consider the corresponding allocation ω^N derived from the equilibrium outcome of the separating equilibrium (s_1^N, s_2^N) for the realization σ^N.[16] For that same type-configuration and for a known state z, consider an alternative N-agent economy in which each agent has the utility function $u(q, r, ; x, z)$, but the endowment for each active agent is $(\bar{q} + 2\kappa_q, \bar{r} + 2\kappa_r)$, while the endowment for each inactive agent is $(0, 0)$. Let ω'' be the competitive allocation for this 2κ-economy, which is unique. By contradiction, let ω' be (1.5κ)-feasible and ex post ε pareto superior to ω^N. Also, let $\omega'' = (\omega_0'', \omega_1'')$ and $\omega^N = (\omega_0^N, \omega_1^N)$, where in each case the subscript 0 is the part of the allocation that pertains to inactive agents and the subscript 1 to the part which pertains to active agents. By definition, $\omega_0'' \equiv (0, 0)$.

By our construction of $Y(z, c^N)$, both ω_1'' and ω_1^N approach the same limit, the competitive allocation for $\mathcal{L}^z$, as $(\kappa, N) \to (0, \infty)$. Therefore, by (15), for N sufficiently large and κ sufficiently small, ω' is pareto superior to ω''. (This uses $\omega_0'' \equiv (0, 0)$.) However, for small enough η, ω'' uses more resources than ω'. This contradicts the first welfare theorem; an economy with fewer resources and the same strictly increasing utility functions cannot have a feasible allocation that is pareto superior to ω'', which is the competitive allocation in the 2κ-economy.

6 Concluding remarks

As noted above, our mechanism violates feasibility. The payoffs of inactive agents, which are determined by the execution of their stage-1 offers at the exogenous price, $\bar{r}/\bar{q}$, and the exogenous stage-2 offers, κ, give rise to a net payout of one of the goods. Feasibility could be restored using entry fees levied on all agents before types are realized. In particular, if the entry fee is 2κ, then η can be chosen to insure feasibility. And, provided there is sufficient motivation for trade coming from the appearance of types in the utility function, κ can be chosen to be small enough to induce participation. With such an entry fee, we could impose $\delta = 0$ in our notion of almost ex post efficiency (and in Theorem 3). That would allow us to achieve the same kind of efficiency result as those in Gul and Postlewaite [6] and in McLean and Postlewaite [8].

A mechanism that would insure feasibility except for κ and would more closely resemble pari-mutuel betting would have the stage-1 offers of the inactive agents be part of the offers that determine the "price" in the second-stage market game and would have their payoffs determined as they are for active agents. However, that would give rise to two-way interaction between the stages. In such a version, if the economy is sufficiently large, it seems as if agents at stage 1 would, as in our version, make stage-1 offers based on

[16] Notice that here ω^N describes a deterministic allocation for the particular realization (σ^N, z, c^N).

the presumption that they will be chosen to be inactive. Even so, they would want to predict the stage-2 price which, itself, is affected by their offers—both directly and by the information revealed by stage-1 offers. Thus, to get an equilibrium, a mapping that takes both stages into account would have to be studied. Moreover, the mapping would have to be defined over all feasible stage-1 actions, including stage-1 actions that give rise to asymmetric information at stage-2. Our approach decouples stage-1 payoffs from what happens at stage 2 and, therefore, is simpler. Given that it has good welfare properties, its simplicity is a virtue—both for us in analyzing the properties of the mechanism and for those who play the game.

We assume a finite number of types and a finite number of states-of-the-world. The latter plays no role. In contrast, the former is important for us. Although the realization of types is random, as the size of the economy grows, conditional independence of types gives us something that resembles replication in a deterministic version. Even more important, our existence result, via the specification of beliefs, depends on a finite number of types.[17] The assumption that all agents have the same endowments plays no role for existence. For existence, this assumption could be replaced by any profile of endowments that is common knowledge. However, for our uniqueness result, if agents' endowments are heterogeneous (or there is another source of known heterogeneity), then assumption A2 would need to be modified: we would require that the second stage be shut down whenever all agents with the same endowment make the same offer at the first stage. Thus, for uniqueness, our detail-freeness claim is somewhat weakened. Also, the size-of-the-economy requirements in our theorems depend on details—the utility function and the information structure.

Finally, regarding the information structure, the two special cases of the model noted at the outset deserve further comment. All our results hold for the pure-private value case in which the state z does not affect preferences. In that case, the state remains a source of aggregate risk because it determines the proportions of agents' types. Therefore, stage-1 remains useful because information aggregation is important for ex post efficiency. In contrast, the pure common-value case, although a special case, is problematic for two reasons related to the *no-trade theorem*. First, our existence result requires that every type trades at the limit economy with $\kappa = 0$, an assumption that depends on the appearance of types in preferences. Second, if types do not appear in preferences, then trade disappears at stage 2 as $\kappa \to 0$ and agents may not want to enter in the presence of an entry fee.

7 Appendix: Proofs

Lemma 1. Fix stage-2 offers of all other agents. Given those offers, for any offer $b' \in [0, \bar{q}] \times [0, \bar{r}]$, there exists $b'' \in \mathcal{O}$ that has the same payoff as b'.

Proof. Let $(Q_-, R_-) \in \mathbb{R}^2_{++}$ be total offers of other agents (including the exogenous

[17] Reny and Perry [11] cannot use a specification with a finite number of types because they have a limit-order mechanism. With such a mechanism and a finite number of types, there can remain an indeterminacy regarding how the gains from trade are distributed. That does not happen for market-game mechanisms.

offers). For any $b \in [0, \bar{q}] \times [0, \bar{r}]$, (3) implies that the corresponding payoffs are

$$q(b_q, b_r) = \bar{q} + \frac{b_r Q_- - b_q R_-}{R_- + b_r} \text{ and } r(b_q, b_r) = \bar{r} + \frac{b_q R_- - b_r Q_-}{Q_- + b_q}. \tag{17}$$

Case (i): $b'_r Q_- - b'_q R_- > 0$. In this case, let $b''_q = 0$ and let b''_r be the unique solution to

$$\frac{b''_r Q_-}{R_- + b''_r} = \frac{b'_r Q_- - b'_q R_-}{R_- + b'_r} \equiv \gamma, \tag{18}$$

where it follows that $\gamma \in (0, Q_-)$. The solution is $b''_r = R_- \gamma / (Q_- - \gamma)$. It follows by (18) that $q(b''_q, b''_r) = q(b'_q, b'_r)$. Also,

$$r(b''_q, b''_r) - \bar{r} = b''_r = R_- \gamma / (Q_- - \gamma) = r(b'_q, b'_r) - \bar{r},$$

where the last equality follows from the definition of γ.

Case (ii): $b'_r Q_- - b'_q R_- < 0$. This is completely analogous, but with $b''_r = 0$.

Case (iii): $b'_r Q_- - b'_q R_- = 0$. Here, of course, we let $b''_q = b''_r = 0$.■

7.1 Existence

Proposition 1. Let $\mathcal{B}$ denote the set of symmetric equilibria for the game $\mathcal{E}(M, \sigma/M, \phi, \kappa)$. If $\kappa > 0$, then $\mathcal{B}$ is not empty.

Proof. Let $S = \{[0, \bar{q}] \times [0, \bar{r}]\}^X$, which is compact and convex. We let $s = \{s^y\}_{y \in X}$ with $s^y = (s^y_q, s^y_r)$ denote a generic element of S. For $s \in S$ and $x \in X$, let $F : S \to S$ be given by

$$F_x(s) = \arg\max_{b \in \mathcal{O}} H_x(b; Q_-, R_-), \tag{19}$$

where

$$H_x(b; Q_-, R_-) = \sum_{z \in Z} \phi(z) u(\bar{q} + \frac{b_r Q_- - b_q R_-}{R_- + b_r}, \bar{r} + \frac{b_q R_- - b_r Q_-}{Q_- + b_q}; x, z) \tag{20}$$

and

$$(Q_-, R_-) = M\kappa + \sum_{y \neq x} \sigma(y) s^y + [\sigma(x) - 1] s^x.$$

Here ϕ is the common posterior on z. We have to show that $F_x(s)$ is unique and is continuous in s. We start with uniqueness. Notice that $(Q_-, R_-) \in \mathbb{R}^2_{++}$ for any $s \in S$.

Because of the $b_q b_r = 0$ constraint in (19), it is helpful to consider $H_x(b_q, 0; Q_-, R_-)$ and $H_x(0, b_r; Q_-, R_-)$ separately, where

$$H_x(b_q, 0; Q_-, R_-) = \sum_{z \in Z} \phi(z) u(\bar{q} - b_q, \bar{r} + \frac{b_q R_-}{Q_- + b_q}; x, z) \equiv g(b_q),$$

and

$$H_x(0, b_r; Q_-, R_-) = \sum_{z \in Z} \phi(z) u(\bar{q} + \frac{b_r Q_-}{R_- + b_r}, \bar{r} - b_r; x, z) \equiv h(b_r).$$

For any $(Q_-, R_-) \in \mathbb{R}^2_{++}$, the functions $f_q(b_q) = \bar{r} + \frac{b_q R_-}{Q_- + b_q}$ and $f_r(b_r) = \bar{q} + \frac{b_r Q_-}{R_- + b_r}$ are strictly concave. Then, because u is strictly concave and because a strictly increasing concave function of a concave function is strictly concave, both g and h are strictly concave. It follows that g has a unique maximum and that h has a unique maximum, denoted $\hat{b}_q$ and $\hat{b}_r$, respectively. Moreover, by the Inada conditions on u, these maxima are characterized by

$$\hat{b}_q = \begin{cases} 0 \text{ if } g'(0) \leq 0 \\ \text{satisfies } g'(\hat{b}_q) = 0 \text{ if } g'(0) > 0 \end{cases}, \tag{21}$$

and

$$\hat{b}_r = \begin{cases} 0 \text{ if } h'(0) \leq 0 \\ \text{satisfies } h'(\hat{b}_r) = 0 \text{ if } h'(0) > 0 \end{cases}. \tag{22}$$

Therefore, a sufficient condition for uniqueness is $\min\{g'(0), h'(0)\} \leq 0$, where

$$g'(0) = \sum_{z \in Z} \phi(z) \left[-u_q(\bar{q}; x, z) + u_r(\bar{r}; x, z) \frac{R_-}{Q_-} \right],$$

and

$$h'(0) = \sum_{z \in Z} \phi(z) \left[u_q(\bar{q}; x, z) \frac{Q_-}{R_-} - u_r(\bar{r}; x, z) \right].$$

Therefore,

$$sign[h'(0)] = sign[\frac{R_-}{Q_-} h'(0)] = sign[-g'(0)] = -sign[g'(0)], \tag{23}$$

which implies $\min\{g'(0), h'(0)\} \leq 0$.

Now we turn to continuity in s, which follows if $(\hat{b}_q, \hat{b}_r)$ is continuous in (Q_-, R_-). By (23), $g'(0) = 0$ *iff* $h'(0) = 0$. That and (21) and (22) imply that $\max\{\hat{b}_q, \hat{b}_r\}$ satisfies a first-order condition with equality. Then, the implicit-function theorem applied to that first-order condition gives the required continuity.

It follows that the mapping F satisfies the hypotheses of Brouwer's fixed-point theorem. Although the domain of the mapping, S, does not satisfy $b_q b_r = 0$, the range does. Therefore, the fixed point satisfies $b_q b_r = 0$.■

Proposition 2. Fix $z \in Z$ and fix a regular CE for $\mathcal{L}^z$ where every type trades, $\langle p^{z,0}, (q_x^{z,0}, r_x^{z,0})_{x \in X} \rangle$. Let $A = [0,1] \times \Delta(X) \times \Delta(Z)$ and let $a_0^z = (0, \mu_z, \delta_z)$, where δ_z is the dirac measure concentrated at z. There exists $\bar{\kappa} = (\bar{\kappa}_q, \bar{\kappa}_r) > 0$ (not dependent on z) such that if $\kappa \in (0, \bar{\kappa}]$,[18] then

[18]Here, $(0, \bar{\kappa}] = (0, \bar{\kappa}_q] \times (0, \bar{\kappa}_r]$.

(i) the economy $\mathcal{L}^z(\kappa)$ has a regular CE, $\langle p^{z,\kappa}, (q_x^{z,\kappa}, r_x^{z,\kappa})_{x\in X}\rangle$, in which every type trades, and whose corresponding offer, denoted $\beta^{z,\kappa}$ (see (5)), is continuous in κ (even at $\kappa = (0,0)$);

(ii) there exists an open neighborhood of a_0^z, denoted $A_{z,\kappa} \subset A$, and a continuous function $f_{z,\kappa} : A_{z,\kappa} \to \mathcal{O}^X$ such that $f_{z,\kappa}(a_0^z) = \beta^{z,\kappa}$ and $(1/M, \sigma/M, \phi^\sigma) \in A_{z,\kappa}$ implies that $f_{z,\kappa}(1/M, \sigma/M, \phi^\sigma)$ is a symmetric equilibrium in $\mathcal{E}(M, \sigma/M, \phi^\sigma, \kappa)$.

Proof. (i) This follows from standard arguments using the Implicit Function Theorem.

(ii) First, we give a claim that is used to evaluate a determinant that appears when we verify a full-rank condition.

Claim 1. Let $\mathbf{a}_n = (a_1, ..., a_n) \in \mathbb{R}^n$, $\mathbf{k}_n = (k_1, ..., k_n) \in \mathbb{R}^n$, and $\mathbf{P}_n = \mathbf{a}_n'\mathbf{k}_n - \mathbf{I}_n$ (where $\mathbf{I}_n$ is the $n \times n$ identity matrix). Then $|\mathbf{P}_n| = (-1)^{n+1}(\sum_{i=1}^n a_i k_i - 1)$.

Proof. The proof is by induction on n. The claim holds for $n = 1$. Now, suppose it holds for n. By definition, $\mathbf{P}_n = [k_1\mathbf{a}_n', k_2\mathbf{a}_n', ..., k_n\mathbf{a}_n'] - \mathbf{I}_n$. If $k_{n+1} = 0$, then $|\mathbf{P}_{n+1}| = -1\,|\mathbf{P}_n|$. Thus, we can assume that $\Pi_{i=1}^{n+1} k_i \neq 0$. Then,

$$|\mathbf{P}_{n+1}| = \left|[k_1\mathbf{a}_{n+1}', k_2\mathbf{a}_{n+1}', ..., k_n\mathbf{a}_{n+1}', k_{n+1}\mathbf{a}_{n+1}'] - \mathbf{I}_n\right| = (\Pi_{i=1}^{n+1} k_i)\left|\begin{pmatrix} \mathbf{A} & \mathbf{b} \\ \mathbf{c} & -\frac{1}{k_{n+1}} \end{pmatrix}\right|,$$

where

$$\mathbf{A} = [\mathbf{a}_n', ..., \mathbf{a}_n'] - \mathrm{diag}(\frac{1}{k_1}, ..., \frac{1}{k_n}), \mathbf{b}' = (1/k_1, 0, ..., 0), \text{ and } \mathbf{c} = (a_{n+1}, ..., a_{n+1}).$$

Then,

$$|\mathbf{P}_{n+1}| = (\Pi_{i=1}^{n+1} k_i)[-(1/k_{n+1})|\mathbf{A}| + (-1)^n(1/k_1]|\mathbf{B}|],$$

where

$$\mathbf{B} = \begin{pmatrix} a_2 & a_2 - 1/k_2 & \dots & a_2 \\ \vdots & \vdots & \ddots & \vdots \\ a_n & a_n & \dots & a_n - 1/k_n \\ a_{n+1} & a_{n+1} & \dots & a_{n+1} \end{pmatrix}$$

and

$$|\mathbf{B}| = \left|\begin{pmatrix} a_2 & -1/k_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ a_n & 0 & \dots & -1/k_n \\ a_{n+1} & 0 & \dots & 0 \end{pmatrix}\right| = (-1)^{n-1} a_{n+1} \left(\Pi_{i=2}^{n+1}(-1/k_i)\right) = a_{n+1}\frac{1}{\Pi_{i=2}^{n+1} k_i}.$$

By the induction hypothesis,

$$|\mathbf{A}| = (-1)^{n+1}\frac{1}{\Pi_{i=1}^n k_i}(\sum_{i=1}^n k_i a_i - 1).$$

Therefore,

$$|\mathbf{P}_{n+1}| = (-1)^{n+2}(\sum_{i=1}^n k_i a_i - 1) + (-1)^n a_{n+1}k_{n+1} = (-1)^{n+2}(\sum_{i=1}^{n+1} k_i a_i - 1).$$

$\square$

Now we prove (ii). Let $\bar{\kappa}$ be given by (i). Fix $\kappa \in (0, \bar{\kappa}]$, and, by (i), let $CE^{z,\kappa} = \langle p^{z,\kappa}, (q_x^{z,\kappa}, r_x^{z,\kappa})_{x \in X} \rangle$ be the constructed regular CE in which every type trades, and let $\beta^{z,\kappa}$ be the corresponding offers given by (5). Hence, $\beta^{z,\kappa}(x) \neq (0,0)$ for all $x \in X$. Let $X_1 = \{x \in X : \beta_q^{z,\kappa}(x) > 0\}$ and let $X_2 = \{x \in X : \beta_r^{z,\kappa}(x) > 0\}$. Then, $X_1 \cap X_2 = \emptyset$ and $X = X_1 \cup X_2$.

The first step is to define the function to which we apply the Implicit Function Theorem. We use β to denote $\langle (\beta_q(x))_{x \in X_1}, (\beta_r(x))_{x \in X_2} \rangle$, the offers, and let $a = (\epsilon, \mu, \phi) \in D \times \Delta(Z)$ be the parameter vector, where

$$D = \{(\epsilon, \mu) \in [-1, 1] \times \Delta(X) : \mu(x) > \epsilon^2 \text{ for all } x \in X\}.$$

(In what follows, $M^{-1/2}$ is one possible magnitude of ϵ.) Let

$$H_x(b_q, b_r; \beta, a) = \sum_{z \in Z} \phi(z) u\left(q, r; x, z\right), \tag{24}$$

where

$$q = \bar{q} + \frac{b_r q_- - b_q r_-}{r_- + \epsilon^2 b_r} \text{ and } r = \bar{r} + \frac{b_q r_- - b_r q_-}{q_- + \epsilon^2 b_q},$$

and

$$(q_-, r_-) = \begin{cases} \left(\sum_{x' \in X_1} \mu(x')\beta_q(x') + \kappa_q - \epsilon^2 \beta_q(x), \sum_{x' \in X_2} \mu(x')\beta_r(x') + \kappa_r\right) \text{ if } x \in X_1 \\ \\ \left(\sum_{x' \in X_1} \mu(x')\beta_q(x') + \kappa_q, \sum_{x' \in X_2} \mu(x')\beta_r(x') + \kappa_r - \epsilon^2 \beta_r(x)\right) \text{ if } x \in X_2 \end{cases}.$$

When $\epsilon^2 = M^{-1}$, H_x is the stage-2 objective function of an agent expressed in terms of average offers of others, (q_-, r_-). Also, because $(\epsilon, \mu) \in D$, $(q_-, r_-) > 0$.

Now, let

$$G_x(\beta, a) = \begin{cases} \arg\max_{b_q \geq 0} H_x(b_q, 0; \beta, a) \text{ if } x \in X_1 \\ \\ \arg\max_{b_r \geq 0} H_x(0, b_r; \beta, a) \text{ if } x \in X_2 \end{cases}.$$

Because each branch of H_x is strictly concave and, hence, has a unique maximum, G_x is a well-defined function. Moreover, by the Theorem of the Maximum, G_x is continuous in all its arguments. Letting $G = (G_x)_{x \in X}$, the function to which we apply the implicit function theorem is $G(\beta, a) - \beta$, whose range is in $\mathbb{R}^X$.

Before doing that, there are several facts about G that we will use. First, let $a_0 = (0, \mu_z, \delta_z)$ and let

$$p(\beta) = \frac{\sum_{y \in X_2} \mu_z(y)\beta_r(y) + \kappa_r}{\sum_{x \in X_1} \mu_z(x)\beta_q(x) + \kappa_q}. \tag{25}$$

Then, for all $x \in X_1$,

$$G_x(\beta, a_0) = \max\{\bar{q} - q_x^z(p(\beta)), 0\}, \tag{26}$$

and for all $y \in X_2$,

$$G_y(\beta, a_0) = \max\{p(\beta)[q_y^z(p(\beta)) - \bar{q}], 0\}, \tag{27}$$

where $q_x^z(p)$ is the demand function of good q for type-x under known state z. Because $\beta^{z,\kappa}$ is a CE in $\mathcal{L}^z(\kappa)$, it follows that $G(\beta^{z,\kappa}, a_0) = \beta^{z,\kappa}$. Notice that a_0 is in the interior of D because $\mu_z(x) > 0$ for all $x \in X$ and for all $z \in Z$. Second, there is an open neighborhood of $(\beta^{z,\kappa}, a_0)$ such that G is continuously differentiable in that neighborhood and has a positive offer of q for all $x \in X_1$ and has a positive offer of r for $x \in X_2$. The proof of this claim follows from the fact that each branch of H_x is strictly concave and continuously differentiable near (β^*, a_0), so that the Implicit Function Theorem can be applied to the first-order conditions that characterize G in that neighborhood. This is where the assumption that all types trade in $\mathcal{L}^z$ is used. Third, if $\sigma : X \to \{0, ..., M\}$ is a type-configuration of M agents and if $\beta \in \mathcal{O}^X$ satisfies $G(\beta, 1/\sqrt{M}, \sigma/M, \phi) = \beta$ with $\beta_q(x) > 0$ for all $x \in X_1$ and $\beta_r(x) > 0$ for all $x \in X_2$, then β is a symmetric equilibrium in $\mathcal{E}(M, \sigma/M, \phi^\sigma, \kappa)$. This follows because H_x defined by (24) is the same as that defined by (20) in the proof of Proposition 1 if $\epsilon^2 = 1/M$ and $\mu = \sigma/M$ and because the strict concavity of each branch of H_x implies that the sign restriction in each branch in (24) is not binding if the maximum is attained at positive offers.

The last preliminary step is to set out the partial derivatives of G w.r.t. to β evaluated at $(\beta, a) = (\beta^{z,\kappa}, a_0)$. Notice that, by (5), $p(\beta^{z,\kappa}) = p^{z,\kappa}$. Using equations (25)-(26), we compute the derivatives according to the chain rule and obtain

$$\frac{\partial G_{x'}(\beta^{z,\kappa}, a_0)}{\partial \beta_q(x)} = \begin{cases} -\frac{\partial}{\partial \beta_q(x)} p(\beta^{z,\kappa}) \frac{d}{dp} q_{x'}^z(p^{z,\kappa}) \text{ for } (x, x') \in X_1 \times X_1 \\ \frac{\partial}{\partial \beta_q(x)} p(\beta^{z,\kappa}) \left[p^{z,\kappa} \frac{d}{dp} q_{x'}^z(p^{z,\kappa}) + (q_{x'}^z(p^{z,\kappa}) - \bar{q}) \right] \text{ for } (x, x') \in X_1 \times X_2 \end{cases}$$

and

$$\frac{\partial G_{x'}(\beta^*, a_0)}{\partial \beta_r(x)} = \begin{cases} \frac{\partial}{\partial \beta_r(x)} p(\beta^{z,\kappa}) \left[p^{z,\kappa} \frac{d}{dp} q_{x'}^z(p^{z,\kappa}) + (q_{x'}^z(p^*) - \bar{q}) \right] \text{ for } (x, x') \in X_2 \times X_2 \\ -\frac{\partial}{\partial \beta_q(x)} p(\beta^{z,\kappa}) \frac{d}{dp} q_{x'}^z(p^{z,\kappa}) \text{ for } (x, x') \in X_2 \times X_1 \end{cases}.$$

Now, let $|X| = L$. Notice that $\left[\frac{\partial G(\beta^{z,\kappa}, a_0)}{\partial \beta(x)}\right]_{x \in X} = \mathbf{a}_L' \mathbf{k}_L$ with

$$\mathbf{a}_L = [(\frac{d}{dp} q_x^z(p^{z,\kappa}))_{x \in X_1}, (\frac{d}{dp} q_{x'}^z(p^{z,\kappa}) + (q_{x'}^z(p^*) - \bar{q}))_{x' \in X_2}]$$

and

$$\mathbf{k}_L = [(-\frac{\partial}{\partial \beta_q(x)} p(\beta^{z,\kappa}))_{x \in X_1}, (\frac{\partial}{\partial \beta_r(x')} p(\beta^{z,\kappa}))_{x' \in X_2}].$$

Thus, by Claim 1, $\left|\left[\frac{\partial G(\beta^{z,\kappa},a_0)}{\partial\beta(x)}\right]_{x\in X}-\mathbf{I}_L\right|=(-1)^{L+1}C$, where

$$\begin{aligned}
C &= \sum_{x\in X_1}(-\frac{\partial p(\beta^{z,\kappa})}{\partial\beta_q(x)})\frac{dq_x^z(p^{z,\kappa})}{dp}+\sum_{y\in X_2}\frac{\partial p(\beta^{z,\kappa})}{\partial\beta_r(y)}\left[p^{z,\kappa}\frac{d}{dp}q_x^z(p^{z,\kappa})+(q_y^z(p^{z,\kappa})-\bar{q})\right]-1\\
&= \frac{p^{z,\kappa}}{\sum_{x\in X_1}\mu_z(x)\beta_q^{z,\kappa}(x)+\kappa_q}\sum_{x\in X_1}\mu_z(x)\frac{dq_x^z(p^{z,\kappa})}{dp}+\frac{(p^{z,\kappa})^2}{\sum_{y\in X_2}\mu_z(y)\beta_r^{z,\kappa}(y)+\kappa_r}\sum_{y\in X_2}\mu_z(y)\frac{dq_y^z(p^{z,\kappa})}{dp}\\
&+ \frac{p^{z,\kappa}}{\sum_{y\in X_2}\mu_z(y)\beta_r(y)+\kappa_r}\sum_{y\in X_2}\mu_z(y)[q_y^z(p^{z,\kappa})-\bar{q}]-1\\
&= \left[\frac{(p^{z,\kappa})^2}{\sum_{y\in X_2}\mu_z(y)\beta_r^{z,\kappa}(y)+\kappa_r}\sum_{x\in X}\mu_z(x)\frac{dq_x^z(p^{z,\kappa})}{dp}\right]+\left[\frac{\sum_{y\in X_2}\mu_z(y)\beta_r(y)}{\sum_{y\in X_2}\mu_z(y)\beta_r(y)+\kappa_r}-1\right]\\
&= \frac{(p^{z,\kappa})^2}{\sum_{y\in X_2}\mu_z(y)\beta_r^{z\kappa}(y)+\kappa_r}\left[\sum_{x\in X}\mu_z(x)\frac{dq_x^z(p^{z,\kappa})}{dp}+\frac{-\kappa_r}{(p^*)^2}\right].
\end{aligned}$$

The last expression differs from zero because the CE corresponding to $\beta^{z,\kappa}$ is regular.

Therefore, by the implicit function theorem, there is a neighborhood $B_{z,\kappa}\subset D\times\Delta(Z)$ around a_0 and a continuously differentiable function $g_{z,\kappa}:B_{z,\kappa}\to([0,\bar{q}]^{X_1}\times[0,\bar{r}]^{X_2})$ such that $g_{z,\kappa}(a_0)=\beta^{z,\kappa}$, and $G(g_{z,\kappa}(a),a)=g_{z,\kappa}(a)$ for all $a\in B_{z,\kappa}$. Because $\beta_q^{z,\kappa}(x)>0$ for all $x\in X_1$ and $\beta_r^{z,\kappa}(y)>0$ for all $y\in X_2$ and because $g_{z,\kappa}$ is continuous, there exists an open neighborhood $A_{z,\kappa}\subset B_{z,\kappa}$ containing a_0 such that for all $a\in A_{z,\kappa}$, $g_{z,\kappa}(a)$ is strictly positive in all its coordinates. Thus, if $(1/\sqrt{M},\sigma/M,\phi^\sigma)\in A_{z,\kappa}$ for a type-configuration $\sigma:\{0,...,M\}\to\mathcal{O}$, then $g_{z,\kappa}(1/\sqrt{M},\sigma/M,\phi^\sigma)$ is a symmetric equilibrium for $\mathcal{E}(M,\sigma/M,\phi^\sigma,\kappa)$. Finally, to deal with the square root, let $f_{z,\kappa}(\epsilon,\mu,\phi)=g_{z,\kappa}(\sqrt{\epsilon},\mu,\phi)$. ■

Theorem 1. Suppose that $\alpha^*(x)\neq\alpha^*(y)$ for any $x\neq y$ and that $\kappa\in(0,\bar{\kappa}]$, where $\bar{\kappa}$ satisfies Proposition 2. There exists $\bar{N}$ such that if $N\geq\bar{N}$, then the N-agent economy has a separating equilibrium.

Proof. Fix $\kappa\in(0,\bar{\kappa}]$, where $\bar{\kappa}$ is from Proposition 2, and let $\langle p^{z,\kappa},(q_x^{z,\kappa},r_x^{z,\kappa})_{x\in X})\rangle$ be a regular CE where every type trades with corresponding offer $\beta^{z,\kappa}$, as constructed in Proposition 2 (i). We show that for large N's, $((s_1^*,s_2^*),\varphi^*)$ is a PBE, where (s_1^*,s_2^*) is given by (11)-(13), and φ^* is given by (9)-(10). Notice that both s_2^* and φ^* depend on N but not on s_1^*. By construction and Proposition 2 (ii), s_2^* is a best response against s_2^* w.r.t. φ^* and φ^* is consistent with Bayes' rule. It remains to show that s_1^* is a best response to (s_1^*,s_2^*) for sufficiently large N.

Let $M^N=\lceil(1-\eta)N\rceil$ be the number of active agents and consider an agent of type x. Because the assignment into active/inactive categories is drawn independently from the types, conditional on being active, the agent's belief about other agents' types is such that those types are i.i.d. with marginal probabilities $(\mu_z(x))_{x\in X}$ conditional on each state z. Let γ_z^N be the i.i.d. distribution over X^{M^N-1} generated by $(\mu_z(x))_{x\in X}$. Given s_2^*, the first-stage problem for the agent of type x is $\max_{a\in\mathcal{O}}G_x^N(a)$, where

$$G_x^N(a)=\eta G_x(a)+(1-\eta)F_x^N(a). \tag{28}$$

Here, G_x is the stage-1 problem contingent on being inactive, and $F_x^N(a)$ is the expected payoff contingent on being active and playing offer a at stage 1.

Claim 1. Let $q_1(a;x)$ be the consumption of q of a type-x agent who plays a and becomes inactive. There exists $\epsilon > 0$ such that if $q_1(a;x) \notin I(x)$, then $G_x(a) < G_x(s_1^*(x)) - \epsilon$.

Proof. As mentioned before, $\max_{a\in\mathcal{O}} G_x(a)$ is equivalent to $\max_{q\in[0,2\bar{q}]} L_x(q)$, where

$$L_x(q) = \sum_{z\in Z} \tau_x(z) u(q, p_1\bar{q} + \bar{r} - p_1 q; x, z).$$

Let $2\delta_x = \min_{y\in X,\ y\neq x} |q_1(\alpha^*(x)) - q_1(\alpha^*(y))|$. Then, $q \notin I(x)$ implies $|q - q_1(\alpha^*(x))| \geq \delta_x$. Because $L_x(q)$ is strictly concave in q and has a maximum at $q_1(\alpha^*(x))$, it follows that $A_x = \min\{-L_x'(q_1(\alpha^*(x)) + \frac{\delta_x}{2}), L_x'(q_1(\alpha^*(x)) - \frac{\delta_x}{2})\} > 0$. Then, for any q such that $|q - q_1(\alpha^*(x))| \geq \frac{\delta_x}{2}$, $L_x(q) \leq L_x(q_1(\alpha^*(x))) - \frac{\delta_x}{2} A_x$. Take $\epsilon_x = (\delta_x/4)A_x$. Then, $q_1(a;x) \notin I(x)$ implies $G_x(a) = L_x(q_1(a;x)) \leq L_x(q_1(\alpha^*(x))) - 2\epsilon_x < G_x(s_1^*(x)) - \epsilon_x$. Finally, let $\epsilon = \min\{\epsilon_x\}_{x\in X}$.□

Claim 2. For all $a \in \mathcal{O}$,

$$\lim_{N\to\infty} F_x^N(a) = \sum_{z\in Z} \tau_x(z) u(q_x^{z,\kappa}, r_x^{z,\kappa}; x, z), \tag{29}$$

uniformly over $\mathcal{O}$. (Recall that $(q_x^{z,\kappa}, r_x^{z,\kappa})_{x\in X}$ is the regular CE allocation of $\mathcal{L}^z(\kappa)$ that (11) uses to construct the strategy profiles β^σ).

Proof. First we give explicit expressions for $F_x^N(a)$. For each $a \in I(\bar{x})$,

$$F_x^N(a) = \sum_{z\in Z} \tau_x(z) \left[\sum_{\xi\in X^{M^N-1}} \gamma_z^N(\xi) \left[u(q^N(a;z,\xi), r^N(a;z,\xi); x, z) \right] \right], \tag{30}$$

where for each z and $\xi = (\xi_1, ..., \xi_{M^N-1}) \in X^{M^N-1}$, the types of the other active agents.

$$q^N(a;z,\xi) = \bar{q} + \frac{s_{2,r}^*(x,a,\nu^{\xi,-a})Q_-^N - s_{2,q}^*(x,a,\nu^{\xi,-a})R_-^N}{s_{2,r}^*(x,a,\nu^{\xi,-a}) + R_-^N},$$

and

$$r^N(a;z,\xi) = \bar{r} + \frac{s_{2,q}^*(x,a,\nu^{\xi,-a})R_-^N - s_{2,r}^*(x,a,\nu^{\xi,-a})Q_-^N}{s_{2,q}^*(x,a,\nu^{\xi,-a}) + Q_-^N}.$$

Here, $\nu^{\xi,-a}$ is the announced histogram given that other active agents' types are ξ and that other agents follow s_1^*, and Q_-^N and R_-^N are the implied stage-2 offers of other active agents according to the candidate equilibrium. That is,

$$\nu^{\xi,-a}(s_1^*(y)) = \sum_{i=1}^{M^N-1} \mathbf{1}_y(\xi_i) \text{ for each } y \in X \text{ and } \nu^{\xi,-a}(a') = 0 \text{ otherwise}, \tag{31}$$

(where $\mathbf{1}_y(y) = 1$ and $\mathbf{1}_y(x) = 0$ if $x \neq y$), and

$$(Q_-^N, R_-^N) = \sum_{y\in X} \sigma^\xi(y)(\beta^{\sigma^\xi}(y) + \kappa) - \beta^{\sigma^\xi}(\bar{x}), \tag{32}$$

where σ^{ξ} is the type-configuration believed by other active agents; namely (recall that $a \in I(\bar{x})$),

$$\sigma^{\xi}(y) = \sum_{i=1}^{M^N-1} \mathbf{1}_y(\xi_i) \text{ for each } y \neq \bar{x} \text{ and } \sigma^{\xi}(\bar{x}) = \sum_{i=1}^{M^N-1} \mathbf{1}_{\bar{x}}(\xi_i) + 1. \tag{33}$$

We prove the claim by showing that, for any infinite sequence of X-valued random variables that is i.i.d. w.r.t. the marginal distribution $(\mu_z(x))_{x\in X}$, $\xi = (\xi_1, ..., \xi_n, ...)$, in which $\xi^{M^N-1} = (\xi_1, ..., \xi_{M^N-1})$ describes the types of the other active agents when there are M^N of them, we have

$$\lim_{N\to\infty} q^N(a; z, \xi^{M^N-1}) = q_x^{z,\kappa}, \quad \lim_{N\to\infty} r^N(a; z, \xi^{M^N-1}) = r_x^{z,\kappa}, \tag{34}$$

almost surely conditional on the state z. Because u is continuous, the claim follows immediately from (30) and (34).

By our construction of off-equilibrium beliefs, (9) and (10), $F_x^N(a)$ depends only on the interval $I(\overline{x})$ such that $a \in I(\overline{x})$. Because there are only finitely many such intervals, uniformity follows from convergence; namely, (29).

By definition, $\xi^{M^N-1} = (\xi_1, ..., \xi_{M^N-1})$ is distributed according to γ_z^N. For each N, let $\sigma^N = \sigma^{\xi^{M^N-1}}$ as defined in (33) (recall that $a \in I(\overline{x})$) and let $\nu^N = \nu^{\xi^{M^N-1},-a}$, as defined in (31). That is, σ^N is the type-configuration believed by all other agents. Then, the sequence $\{\sigma^N\}$ is such that $\sum_{y\in X} \sigma^N(y) = M^N$ and for each $y \in X$, $\lim_{N\to\infty}(\sigma^N(y)/M^N) = \mu_z(y)$ almost surely. Consider a realization of ξ for which $\lim_{N\to\infty}(\sigma^N(y)/M^N) = \mu_z(y)$. Then, for N sufficiently large, $(1/M^N, \sigma^N/M^N, \phi^{\sigma^N}) \in A_z$ and hence, for such N's, $\beta^{\sigma^N} = f_z(1/M^N, \sigma^N/M^N, \phi^{\sigma^N})$. Notice that $\lim_{N\to\infty} \phi^{\sigma^N} = \delta_z$. Thus, by Proposition 2 (ii), we have $\lim_{N\to\infty} \beta^{\sigma^N} = \beta^{z,\kappa}$ (the offers corresponding to the CE $\langle p^{z,\kappa}, (q_x^{z,\kappa}, r_x^{z,\kappa})_{x\in X}\rangle$). This implies that

$$\lim_{N\to\infty} (\frac{Q_-^N}{M^N}, \frac{R_-^N}{M^N}) = \sum_{y\in X} \mu_z(y)(\beta^z(y) + \kappa) \text{ and } \lim_{N\to\infty} \frac{R_-^N}{Q_-^N} = p^z, \tag{35}$$

where Q_-^N and R_-^N are defined in (32) with $\xi = \xi^{M^N-1}$.

Finally, we show that $\lim_{N\to\infty} s_2^*(x, a, \nu^N) = \beta^{z,\kappa}(x)$, where s_2^* is defined in (13). Letting $\phi^N = \text{marg}_Z \varphi^*(x, a, \nu^N)$, where φ^* is defined in (9) and (10), we have

$$\lim_{N\to\infty} \phi^N[z] = 1.$$

Notice that ϕ^N is derived from the type-configuration believed by the agent, which is different from σ^N if $x \neq \bar{x}$. For each N, $s_2^*(x, a, \nu^N)$ solves

$$\max_{b\in\mathcal{O}} H_x^N(b) = \max_{b\in\mathcal{O}} \sum_{z'\in Z} \phi^N[z']u\left(\bar{q} + \frac{b_r Q_-^N - b_q R_-^N}{R_-^N + b_r}, \bar{r} + \frac{b_q R_-^N - b_r Q_-^N}{Q_-^N + b_q}; x, z'\right). \tag{36}$$

Now, let

$$J_x(b; p, c_1, c_2, \phi) = \sum_{z' \in Z} \phi[z'] u\left(q, r; x, z'\right)$$

with $q = \bar{q} + \frac{b_r}{p(1+c_2 b_r)} - \frac{b_q}{1+c_2 b_r}$ and $r = \bar{r} - \frac{b_r}{1+c_1 b_q} + \frac{p b_q}{1+c_1 b_q}$, and where the domain for J_x is $\mathcal{O} \times \left[\frac{\kappa_r}{\bar{q}+\kappa_q}, \frac{\bar{r}+\kappa_r}{\kappa_q}\right] \times \left[0, \frac{1}{\kappa_r}\right] \times \left[0, \frac{1}{\kappa_q}\right] \times \Delta(Z)$. It follows that $J_x(b; \frac{R_-^N}{Q_-^N}, 1/Q_-^N, 1/R_-^N, \phi^N) = H_x^N(b)$. Therefore, by the argument used in the proof of Proposition 1, $J_x(\cdot; p, c_1, c_2, \phi)$ has a unique maximum, $j_x(p, c_1, c_2, \phi)$. And because J_x is continuous on its domain, the Maximum Theorem implies that $j_x(p, c_1, c_2, \phi)$ is continuous.

Now, for each N, $s_2^*(x, a, \nu^N) = j_x(\frac{R_-^N}{Q_-^N}, 1/Q_-^N, 1/R_-^N, \phi^N)$. By (35) and the continuity of j_x, it follows that

$$b^* = \lim_{N\to\infty} s_2^*(x, a, \nu^N) = \lim_{N\to\infty} j_x\left(\frac{R_-^N}{Q_-^N}, 1/Q_-^N, 1/R_-^N, \phi^N\right) = j_x(p^z, 0, 0, \delta_z).$$

By the definition of J_x, it follows that b^* maximizes $u\left(\bar{q} - b_q + \frac{b_r}{p^z}, \bar{r} - b_r + p^z b_q; x, z\right)$. Therefore, b^* is the offer for type-x agents corresponding to the CE, $\langle p^{z,\kappa}, (q_x^{z,\kappa}, r_x^{z,\kappa})_{x\in X}\rangle$; that is, $b^* = \beta^{z,\kappa}(x)$. This shows that

$$\lim_{N\to\infty} q^N(a; z, \xi^{M^N-1}) = \bar{q} + \frac{\beta_r^{z,\kappa}(x)}{p^{z,\kappa}} - \beta_q^{z,\kappa}(x), \quad \lim_{N\to\infty} r^N(a; z, \xi^{M^N-1}) = \bar{r} + \beta_q^{z,\kappa}(x) p^{z,\kappa} - \beta_r^{z,\kappa}(x),$$

almost surely for all $a \in \mathcal{O}$. This proves (34). $\square$

In order to have any effect on $F_x^N(a)$, the agent must choose an offer sufficiently far from s_1^*, the offer that maximizes $G_x(a)$. Claim 1 shows that the implied loss in terms of $G_x(a)$ is bounded away from zero (and does not depend on N). By Claim 2, any effect on $F_x^N(a)$ goes to zero as $N \to \infty$. Together, they imply that s_1^* is a best response to (s_1^*, s_2^*) for sufficiently large N. $\blacksquare$

7.2 Uniqueness

Theorem 2. Suppose that $\alpha^*(x) \neq \alpha^*(y)$ for any $x \neq y$, that the competitive demand for good-q is monotone in $\mathcal{L}^z$ for each $z \in Z$, and that A1-A3 hold. Fix $\kappa > 0$. There exists $\bar{N}$ such that if $N > \bar{N}$, then any equilibrium $s^N = (s_1^N, s_2^N)$ is separating.

Proof. First, we have uniqueness of CE in $\mathcal{L}^z(\kappa)$ for any $\kappa \geq 0$. (Excess demand for good-q in $\mathcal{L}^z(\kappa)$ is $f(p; \kappa) = \sum_{x\in X} \mu_z(x) q_x(p) + \kappa_r/p - \bar{q} - \kappa_q$. Therefore, $\partial f(p; \kappa)/\partial p = \sum_{x\in X} \mu_z(x) \left[\partial q_x(p)/\partial p\right] - \kappa_r/p^2$. Monotonicity of demand functions implies that the first term is negative. Hence, $\partial f(p; \kappa) < 0$, which rules out multiple CE's.) Now, fix some $\kappa > 0$. For each $z \in Z$, let $\beta^{z,\kappa}$ be the offer corresponding to the unique CE in $\mathcal{L}^z(\kappa)$.

First we exclude complete pooling, i.e., an equilibrium s such that for some $\bar{a} \in \mathcal{O}$, $s_1(x) = \bar{a}$ for all $x \in X$.

Claim 0. For any equilibrium s, there exist $x \neq y \in X$ such that $s_1(x) \neq s_1(y)$.

Proof. By way of contradiction, suppose that s is an equilibrium with $s_1(x) = \bar{a}$ for all $x \in X$. By A2, this implies that the realized payoff of all active agents is $(\bar{q}, \bar{r})$, no-trade. However, because α^* is separating, there exists some x such that $G_x(\bar{a}) < G_x(\alpha^*(x))$. Therefore, this agent has a profitable deviation to $\alpha^*(x)$ because no-trade is feasible at stage-2 contingent on being active. □

Claim 0 implies that any candidate equilibrium that is not separating is associated with a partition $\mathcal{Y} = (Y_1, ..., Y_K)$ of X with $1 < K < |X|$. We denote such a candidate equilibrium for N agents by s^N. We prove, by way of contradiction, that s^N cannot be an equilibrium for sufficiently large N. The contradiction is that one agent, called the *potential defector*, has a profitable deviation (to the stage-1 action described by α^*).

For a potential defector of type-x, the stage-1 objective function is

$$G_x^N(a) = \eta G_x(a) + (1 - \eta) F_x^N(a),$$

where $F_x^N(a)$ is the expected payoff implied by offer a at stage-1 conditional on being active. We show that for any $\epsilon > 0$, $F_x^N(a)$ does not vary with a by more than ϵ for sufficiently large N. By assumption A3, $F_x^N(a)$ depends on the associated set of types, $A \subseteq X$, that is believed by other active agents when a is offered at stage-1. Indeed, this holds for equilibrium beliefs as well: if a is an equilibrium offer with $a = s_1^N(y)$, then a is associated with $A = Y_k$ with $y \in Y_k$. Therefore, we may rewrite $F_x^N(a)$ as $F_x^N(A)$ with $A \subseteq X$ being the set of types associated with a. To calculate $F_x^N(A)$, we need to characterize the stage-2 offers following a public announcement ν_N such that $\nu_N(\tilde{a}) = 1$ for some $\tilde{a}$ that is associated with A and $\nu_N(a) = 0$ if $a \notin \{s_1^N(Y_k) : k = 1, ..., K\} \cup \{\tilde{a}\}$.

We divide the rest of the proof into four claims and a final argument. Each of the first three claims has two similar parts—one part for the potential defector and the other for agents following equilibrium behavior. Claim 1 is concerned with beliefs along the equilibrium path and has nothing to do with behavior. Claim 2 provides bounds on offers that assure that consumption of each good is bounded away from zero—bounds that hold in any equilibrium. Those bounds imply bounds on the derivatives that appear in the first-order conditions that hold at all best responses. Claim 3 establishes uniform convergence of equilibrium offers, β^N, to "price-taking" offers with a known state-of-the-world and with a price that is given by the equilibrium offers of others, where the uniformity is over all possible sequences of equilibria. Claim 4 is closely related to Claim 2 in the proof of Theorem 1 because it says that $F_x^N(a)$ does not vary much with a for sufficiently large N. The final argument follows the logic of the proof of Theorem 1. In what follows, let $M^N = \lceil (1 - \eta) N \rceil$ be the number of active agents.

Claim 1. Let ν_N be the public announcement which includes the offer $\tilde{a}^N$ (made by the potential defector), and let λ^N be the corresponding signal configuration for agents other than the potential defector.

(a) Denote the stage-2 belief of the potential defector of type x, following λ^N, by

$$\varphi^N(x, \tilde{a}, \nu_N^{-\tilde{a}})[z, \xi^1, ..., \xi^K] = \tilde{\phi}_x^N[z] \tilde{\gamma}_z^N[\xi^1, \xi^2, ..., \xi^K],$$

where $\tilde{\phi}_x^N[\bar{z}]$ is the posterior over states, $\tilde{\gamma}_z^N$ is that over the types of other active agents conditional on z, $\xi^k = (\xi_1^k, ..., \xi_{\lambda^N(Y_k)}^k) \in Y_k^{\lambda^N(Y_k)}$ describes the types for those agents who made the offer associated with Y_k. Then, for any $\epsilon > 0$, there exist $N_a^1(\epsilon)$ and $\delta_a^1(\epsilon) \leq \epsilon$ such that if $N > N_a^1(\epsilon)$ and if $\left|\frac{\lambda^N(Y_k)}{M^N} - \mu_{z^*}(Y_k)\right| < \delta_a^1(\epsilon)$ for all k, then for each $y \in X$,

$$\tilde{\phi}_x^N[z^*] > 1-\epsilon \text{ and } \tilde{\gamma}_{z^*}^N\left[\left|\tilde{\rho}^N(y) - \frac{\mu_{z^*}(y)}{\mu_{z^*}(Y_k)}\right| < \epsilon\right] > 1-\epsilon,$$

where

$$\tilde{\rho}^N(y) = \#\{\xi_i^k : \xi_i^k = y, i = 1, ..., \lambda^N(Y_k)\}/\lambda^N(Y_k),$$

the fraction of agents other than the potential defector with types in Y_k who are of type-y.

(b) Consider an agent of type x other than the potential defector. For such an agent the relevant signal configuration is the offer $\tilde{a}^N$ (made by the potential defector) and λ_-^N defined as follows: If $x \in Y_{\bar{k}}$, then $\lambda_-^N(Y_k) = \lambda^N(Y_k)$ for each $k \neq \bar{k}$ and $\lambda_-^N(Y_{\bar{k}}) = \lambda^N(Y_{\bar{k}}) - 1$. We denote such an agent's stage-2 belief, after observing the signal configuration λ_-^N and $\tilde{a}^N$, by

$$\varphi^N(y, s_1(y), \nu_N^{-s_1(y)})[z, \xi^1, ..., \xi^K, \tilde{\xi}] = \hat{\phi}_y^N[z]\hat{\gamma}_z^N[\xi^1, ..., \xi^K, \tilde{\xi}],$$

where $\hat{\phi}_x^N$ is the posterior distribution over states, $\hat{\gamma}_z^N$ is that over types of the other active agents conditional on z. Then, for any $\epsilon > 0$, there exist $N_b^1(\epsilon)$ and $\delta_b^1(\epsilon) \leq \epsilon$ such that if $N > N_b^1(\epsilon)$ and if $\left|\frac{\lambda^N(Y_k)}{M^N} - \mu_{z^*}(Y_k)\right| < \delta_b^1(\epsilon)$ for all k, then for each $y \in X$,

$$\hat{\phi}_x^N[z^*] > 1-\epsilon \text{ and } \hat{\gamma}_{z^*}^N\left[\left|\hat{\rho}^N(y) - \frac{\mu_{z^*}(y)}{\mu_{z^*}(Y_k)}\right| < \epsilon\right] > 1-\epsilon,$$

where

$$\hat{\rho}^N(y) = \#\{\xi_i^k : \xi_i^k = y, i = 1, ..., \lambda_-^N(Y_k)\}/\lambda_-^N(Y_k),$$

the fraction of *other* active (non-defecting) agents with types in Y_k who are of type-y.

Proof. By Bayes' rule, we can derive $\hat{\phi}_x^N$, $\hat{\gamma}_z^N$, $\tilde{\phi}_x^N$, and $\tilde{\gamma}_z^N$ as follows:

$$\hat{\phi}_x^N[\bar{z}] = \frac{\pi(\bar{z})\mu_{\bar{z}}(x)\mu_{\bar{z}}(A)\prod_{k=1}^K[\mu_{\bar{z}}(Y_k)]^{\lambda_-^N(Y_k)}}{\sum_{z\in Z}\pi(z)\mu_z(x)\mu_z(A)\prod_{k=1}^K[\mu_z(Y_k)]^{\lambda_-^N(Y_k)}},$$

(recall that A is associated with the offer $\tilde{a}^N$) and

$$\hat{\gamma}_z^N[\xi^1, ..., \xi^K, \tilde{\xi}] = \frac{\mu_z(\tilde{\xi})}{\mu_z(A)}\prod_{k=1}^K\left[\prod_{i=1}^{\lambda_-^N(Y_k)}\left(\frac{\mu_z(\xi_i^k)}{\mu_z(Y_k)}\right)\right],$$

where $\xi^k = (\xi_1^k, ..., \xi_{\lambda^N(Y_k)}^k) \in Y_k^{\lambda^N(Y_k)}$ describes the types for those who make the offer associated with Y_k and $\tilde{\xi}$ describes the type of the agent who offered $\tilde{a}^N$.

Similarly,

$$\tilde{\phi}_x^N[\bar{z}] = \frac{\pi(\bar{z})\mu_{\bar{z}}(x)\prod_{k=1}^K[\mu_{\bar{z}}(Y_k)]^{\lambda^N(Y_k)}}{\sum_{z\in Z}\pi(z)\mu_z(x)\prod_{k=1}^K[\mu_z(Y_k)]^{\lambda^N(Y_k)}},\ \tilde{\gamma}_z^N[\xi^1, ..., \xi^K] = \prod_{k=1}^K\prod_{i=1}^{\lambda^N(Y_k)}\frac{\mu_z(\xi_i^k)}{\mu_z(Y_k)},$$

where $\xi^k = (\xi_1^k, ..., \xi_{\lambda^N(Y_k)}^k) \in Y_k^{\lambda^N(Y_k)}$ describes the types for those agents other than the potential defector who make the offer associated with Y_k ($\lambda^N(Y_k)$ is the number of such agents).

The convergence result for $\hat{\phi}_x^N$ and $\tilde{\phi}_x^N$ follows immediately from the above expressions. We prove only Claim 1(b) for $\hat{\rho}^N$. (The proof for 1(a) is similar.) For each $z \in Z$, consider K infinite sequences of random variables $(\zeta^1, \zeta^2, ..., \zeta^K)$ such that ζ_i^k is Y_k-valued for all $i \in \mathbb{N}$, the K sequences are independent of each other, and ζ^k is an i.i.d. sequence with marginal distribution $(\frac{\mu_z(y)}{\mu_z(Y_k)})_{y \in Y_k}$. Let γ_z denote the joint distribution of $(\zeta^1, \zeta^2, ..., \zeta^K)$. Then, given a sequence of signal-configurations λ^N and a realization $(\zeta^1, \zeta^2, ..., \zeta^K)$, for each $k = 1, ..., K$, each $y \in Y_k$, and each N, define

$$\rho^N(y) = \#\{\zeta_i^k : \zeta_i^k = y, i = 1, ..., \lambda_-^N(Y_k)\}/\lambda_-^N(Y_k).$$

Notice that for each $y \in X$, $\rho^N(y)$ and $\hat{\rho}^N(y)$ have the same distribution. By the *law of large numbers*, for each $y \in Y_k$, $\rho^N(y)$ converges to $\mu_z(y)/\mu_z(Y_k)$ in probability under γ_z for any k as $\lambda_-^N(Y_k)$ converges to infinity. This implies the result.□

Now we turn to equilibrium behavior, where, again, we distinguish between the potential defector and the other agents. We use $\tilde{\beta}^{N,A}(x)$ to denote the equilibrium offer from the potential defector with type x and use $\hat{\beta}^{N,A}(x)$ to denote the equilibrium offer from an agent other than the potential defector with type x.

Claim 2. There exist $\bar{b} = (\bar{b}_q, \bar{b}_r) < (\bar{q}, \bar{r})$ such that $\tilde{\beta}^{N,A}(x) \leq \bar{b}$ and $\hat{\beta}^{N,A}(x) \leq \bar{b}$.

Proof. As might be expected, this follows from the Inada conditions and the bounds on prices implied by $\kappa > 0$. First notice that $\tilde{\beta}^{N,A}(x)$ and $\hat{\beta}^{N,A}(x)$ solve the following problems.

Following a signal configuration λ^N for agents other than the potential defector, the potential defector of type x has the stage-2 objective function,

$$\tilde{H}_x^{\lambda^N}(b_q, b_r) = \sum_{z \in Z} \tilde{\phi}_x^N[z] \mathbb{E}_{\tilde{\gamma}_z^N} \left[u \left(\bar{q} + \frac{b_r \tilde{Q}_-^N - b_q \tilde{R}_-^N}{\tilde{R}_-^N + b_r}, \bar{r} + \frac{b_q \tilde{R}_-^N - b_r \tilde{Q}_-^N}{\tilde{Q}_-^N + b_q}; x, z \right) \right], \tag{37}$$

where

$$(\tilde{Q}_-^N, \tilde{R}_-^N) = \sum_{k=1,..,K} \sum_{y \in Y_k} \lambda^N(Y_k) \tilde{\rho}^N(y) \hat{\beta}^{N,A}(y) + M^N \kappa.$$

Following a signal configuration λ_-^N of other (non-defecting) agents, an agent other than the potential defector of type x has the stage-2 objective function,

$$\hat{H}_x^{\lambda^N}(b_q, b_r) = \sum_{z \in Z} \hat{\phi}_x^N[z] \mathbb{E}_{\hat{\gamma}_z^N} \left[u \left(\bar{q} + \frac{b_r \hat{Q}_-^N - b_q \hat{R}_-^N}{\hat{R}_-^N + b_r}, \bar{r} + \frac{b_q \hat{R}_-^N - b_r \hat{Q}_-^N}{\hat{Q}_-^N + b_q}; x, z \right) \right], \tag{38}$$

where

$$(\hat{Q}_-^N, \hat{R}_-^N) = \sum_{k=1,..,K} \sum_{y \in Y_k} \lambda_-^N(Y_k) \hat{\rho}^N(y) \hat{\beta}^{N,A}(y) + \tilde{\beta}^{N,A}(\tilde{\xi}) + M^N \kappa.$$

Notice that here $\tilde{\xi}$ denotes the potential defector's type.

We prove the claim for $\tilde{\beta}^{N,A}$; the other case is exactly the same. Obviously, we are only concerned with positive offers. We spell out the details for $\tilde{\beta}_q^{N,A}(x) > 0$. To abbreviate notation denote $\tilde{\beta}_q^{N,A}(x)$ by b_q^*.

Being positive, b_q^* satisfies the first-order condition (u_q and u_r denote the first-order derivatives of u),

$$\sum_{z\in Z} \tilde{\phi}_x^N[z] \left\{ \mathbb{E}_{\tilde{\gamma}_z^N} \left[-u_q(q(b_q^*), r(b_q^*); x, z) + u_r(q(b_q^*), r(b_q^*); x, z) \frac{\tilde{Q}_-^N \tilde{R}_-^N}{(\tilde{Q}_-^N + b_q)^2} \right] \right\} = 0 \tag{39}$$

where

$$(q(b_q), r(b_q)) = \left(\bar{q} - b_q, \bar{r} + \frac{b_q \tilde{R}_-^N}{\tilde{Q}_-^N + b_q} \right). \tag{40}$$

For any $b_q \in [0, \bar{q}]$, $\tilde{Q}_-^N \in [M^N \kappa_q, (M^N - 1)\bar{q} + M^N \kappa_q)]$, and $\tilde{R}_-^N \in [M^N \kappa_r, (M^N - 1)\bar{r} + M^N \kappa_r)]$,

$$u_r(q(b_q), r(b_q); x, z) \frac{\tilde{Q}_-^N \tilde{R}_-^N}{(\tilde{Q}_-^N + b_q)^2} \leq \max_{q\in[0,\bar{q}], z\in Z} u_r(q, \bar{r}; x, z) \frac{(\bar{q} + \kappa_q)(\bar{r} + \kappa_r)}{\kappa_q^2} \equiv A.$$

For each $b_q \in [0, \bar{q})$, let

$$J(b_q) = \min_{r\in[\bar{r}, \bar{r} + \frac{\bar{q}(\bar{r}+\kappa_r)}{\kappa_q}], z\in Z} u_q(\bar{q} - b_q, r; x, z).$$

Because $[\bar{r}, \bar{r} + \frac{\bar{q}(\bar{r}+\kappa_r)}{\kappa_q}]$ is compact and X and Z are both finite, J is well-defined, positive, strictly increasing, $\lim_{b_q\to\bar{q}} J(b_q) = \infty$, and, of course, $J(b_q) \leq u_q(q(b_q), r(b_q); x, z)$.

Let $\gamma > 1$ be such that there is a solution for b_q to $J(b_q) = \gamma A$. Denote the solution, which is unique, $\tilde{b}_q(x)$. We next show that $\tilde{\beta}_q^{N,A}(x) = b_q^* \leq \tilde{b}_q(x) < \bar{q}$. The second inequality follows from $\gamma A < \infty$. Suppose the first inequality does not hold. Then, by (39), for some $(z, \tilde{Q}_-^N, \tilde{R}_-^N)$, we must have

$$u_q(q(b_q^*), r(b_q^*); x, z) - u_r(q(b_q^*), r(b_q^*); x, z) \frac{\tilde{Q}_-^N \tilde{R}_-^N}{(\tilde{Q}_-^N + b_q)^2} \leq 0$$

with $(q(b_q), r(b_q))$ as in (40). Because $b_q^* > \tilde{b}_q(x)$,

$$u_q(q(b_q^*)), r(b_q^*); x, z) > \gamma A \text{ and } u_r(q(b_q^*), r(b_q^*); x, z) \frac{\tilde{Q}_-^N \tilde{R}_-^N}{(\tilde{Q}_-^N + b_q)^2} \leq \gamma A,$$

a contradiction. The argument for $\tilde{\beta}_r^{N,A}(x) > 0$ is exactly analogous.

Finally, take $\bar{b}_q = \max\{\tilde{b}_q(x), \hat{b}_q(x) : x \in X\}$ and $\bar{b}_r = \max\{\tilde{b}_r(x), \hat{b}_r(x) : x \in X\}$, where $\hat{b}_q(x)$ and $\hat{b}_r(x)$ are the analogous bounds for $\hat{\beta}^{N,A}(x)$. $\square$

Claim 3. Recall that $\tilde{\beta}^{N,A}$ denotes the equilibrium offer from the potential defector of type-x and $\hat{\beta}^{N,A}$ denotes the equilibrium offer from other agents of type x. Fix a state $z \in Z$. For any $p > 0$ let $\chi(x;p) = (\chi_q(x;p), \chi_r(x;p))$ be the unique solution to

$$\max_{b \in \mathcal{O}} H_x(b;p) = \max_{b \in \mathcal{O}} u(\bar{q} - b_q + \frac{b_r}{p}, \bar{r} - b_r + b_q p; x, z).$$

For any $\epsilon > 0$, there exists $N^3(\epsilon)$ and $\delta^3(\epsilon) \leq \epsilon$ such that if $N > N^3(\epsilon)$ and if $\left|\frac{\lambda^N(Y_k)}{M^N} - \mu_z(Y_k)\right| < \delta^3(\epsilon)$ for all k, then for each $x \in X$,

$$|\tilde{\beta}_q^{N,A}(x) - \chi_q(x;p^N)| + |\tilde{\beta}_r^{N,A}(x) - \chi_r(x;p^N)| < \epsilon$$

and

$$|\hat{\beta}_q^{N,A}(x) - \chi_q(x;p^N)| + |\hat{\beta}_r^{N,A}(x) - \chi_r(x;p^N)| < \epsilon,$$

where

$$p^N = \frac{\sum_{y \in X} \mu_z(y)\hat{\beta}_r^{N,A}(y) + \kappa_r}{\sum_{y \in X} \mu_z(y)\hat{\beta}_q^{N,A}(y) + \kappa_q}.$$

Proof. We first prove the claim for $\tilde{\beta}^{N,A}$. The objective $\tilde{H}_x^{\lambda^N}(b)$, defined in (37), for which $\tilde{\beta}^{N,A}(x)$ is a best response, differs from $H_x(b;p^N)$, for which $\chi(x;p^N)$ is a best response, in two respects. In $H_x(b;p^N)$, offers of others are weighted by limit weights, while in $\tilde{H}_x^{\lambda^N}(b)$ they are weighted by the agent's posterior over the types of others. And, in $H_x(b;p^N)$, the price is unaffected by the agent's own offer, while in $\tilde{H}_x^{\lambda^N}(b)$ it responds to the agent's offer as in the market game. The proof of the claim shows that both differences disappear for sufficiently large N.

Let $d = \frac{1}{2}\min\{\bar{q} - \bar{b}_q, \bar{r} - \bar{b}_r\}$. First we show that, for any $\epsilon > 0$, there exist $N^2(\epsilon)$ and $\delta^2(\epsilon)$ such that if $N > N^2(\epsilon)$ and if $\left|\frac{\lambda^N(Y_k)}{M^N} - \mu_z(Y_k)\right| < \delta^2(\epsilon)$ for all k, then for all $x \in X$ and for all $b_q \in [0, \bar{q} - d]$ and all $b_r \in [0, \bar{r} - d]$,

$$\left|\frac{\partial}{\partial b_q} H_x(b_q, 0; p^N) - \frac{\partial}{\partial b_q}\tilde{H}_x^{\lambda^N}(b_q, 0)\right| < \epsilon \text{ and } \left|\frac{\partial}{\partial b_r} H_x(0, b_r; p^N) - \frac{\partial}{\partial b_r}\tilde{H}_x^{\lambda^N}(0, b_r)\right| < \epsilon. \tag{41}$$

Because the arguments are essentially the same, we only prove the first of these.

Fix some $x \in X$ and let

$$L_z(b_q, p_1, p_2) = u_q(\bar{q} - b_q, \bar{r} + b_q p_1; x, z) - u_r(\bar{q} - b_q, \bar{r} + b_q p_1; x, z)p_2.$$

$L_z(b_q, p_1, p_2)$ is continuous over $[0, \bar{q} - d] \times \left[\frac{\kappa_r}{\bar{q} + \kappa_q}, \frac{\kappa_r + \bar{r}}{\kappa_q}\right]^2$ and, hence, is uniformly continuous. Therefore, for any $\epsilon > 0$, there exists some $\hat{\delta}(\epsilon) \leq \epsilon$ such that

$$|p_1 - p_1'| < \hat{\delta}(\epsilon) \text{ and } |p_2 - p_2'| < \hat{\delta}(\epsilon) \Rightarrow |L_z(b_q, p_1, p_2) - L_z(b_q, p_1', p_2')| < \epsilon. \tag{42}$$

Notice that $\frac{\partial}{\partial b_q} H_x(b_q, 0; p^N) = L_z(b_q, p^N, p^N)$ and that

$$\frac{\partial}{\partial b_q}\tilde{H}_x^{\lambda^N}(b_q, 0) = \sum_{z' \in Z} \tilde{\phi}_x^N[z']\mathbb{E}_{\tilde{\gamma}_{z'}^N}\left[L_{z'}\left(b_q, \frac{\tilde{R}_-^N}{\tilde{Q}_-^N + b_q}, \frac{\tilde{Q}_-^N \tilde{R}_-^N}{(\tilde{Q}_-^N + b_q)^2}\right)\right].$$

Hence, it is sufficient to show that p^N is close to both $\frac{\tilde{R}^N_-}{\tilde{Q}^N_-+b_q}$ and $\frac{\tilde{Q}^N_-\tilde{R}^N_-}{(\tilde{Q}^N_-+b_q)^2}$ as N becomes large and as λ^N/M^N converges uniformly.

Because

$$p^N - \frac{\tilde{R}^N_-}{\tilde{Q}^N_-} = \frac{\sum_{y\in X}\mu_z(y)\hat{\beta}^{N,A}_r(y)+\kappa_r}{\sum_{y\in X}\mu_z(y)\hat{\beta}^{N,A}_q(y)+\kappa_q} - \frac{\sum_{k=1,..,K}\sum_{y\in Y_k}\frac{\lambda^N_-(Y_k)}{M^N}\tilde{\rho}^N(y)\hat{\beta}^{N,A}_r(y)+\kappa_r}{\sum_{k=1,..,K}\sum_{y\in Y_k}\frac{\lambda^N_-(Y_k)}{M^N}\tilde{\rho}^N(y)\hat{\beta}^{N,A}_q(y)+\kappa_q},$$

we have

$$\left|p^N - \frac{\tilde{R}^N_-}{\tilde{Q}^N_-}\right| \le \frac{2(\bar{q}+\kappa_q)(\bar{r}+\kappa_r)}{\kappa_q^2}\sum_{k=1}^{K}\left|\mu_{z^*}(Y_k) - \frac{\lambda^N_-(Y_k)}{M^N}\tilde{\rho}^N(y)\right|.$$

Moreover,

$$\left|\frac{\tilde{Q}^N_-\tilde{R}^N_-}{(\tilde{Q}^N_-+b_q)^2} - \frac{\tilde{R}^N_-}{\tilde{Q}^N_-}\right| \le \frac{\bar{r}+\kappa_r}{\kappa_q}\frac{\bar{q}(2\kappa_q+\frac{\bar{q}}{M^N})}{\kappa_q^2 M^N} \text{ and } \left|\frac{\tilde{R}^N_-}{\tilde{Q}^N_-+b_q} - \frac{\tilde{R}^N_-}{\tilde{Q}^N_-}\right| \le \frac{\bar{r}+\kappa_r}{\kappa_q}\frac{\bar{q}}{\kappa_q M^N}.$$

Hence, for any $\epsilon > 0$ there exist $\tilde{N}(\epsilon)$ and $\tilde{\delta}(\epsilon) \le \epsilon$ such that if $N > \tilde{N}(\epsilon)$ and if $\left|\tilde{\rho}^N(y) - \frac{\mu_z(y)}{\mu_z(Y_k)}\right| < \tilde{\delta}(\epsilon)$ and $\left|\frac{\lambda^N(Y_k)}{M^N} - \mu_z(Y_k)\right| < \tilde{\delta}(\epsilon)$ for all k, then

$$\left|\frac{\tilde{Q}^N_-\tilde{R}^N_-}{(\tilde{Q}^N_-+b_q)^2} - p^N\right| < \epsilon \text{ and } \left|\frac{\tilde{R}^N_-}{\tilde{Q}^N_-+b_q} - p^N\right| < \epsilon \text{ for all } b_q. \tag{43}$$

Let $B = 2\max\{1, u_q(d,r;x,z'), u_r(q,d;x,z') : r \in \left[0, \bar{r}+2\bar{q}\frac{\bar{r}+\kappa_r}{\kappa_q}\right], q \in [0,\bar{q}], z' \in Z\}$. Then, for all $z' \in Z$, $|L_{z'}(b_q,p_1,p_2)| \le \frac{1}{2}B$ for all $(b_q,p_1,p_2) \in [0,\bar{q}-d] \times [\frac{\kappa_r}{\bar{q}+\kappa_q}, \frac{\kappa_r+\bar{r}}{\kappa_q}]^2$. Let $\delta' = \hat{\delta}(\frac{\epsilon}{10B})$ (see (42)). Let

$$\delta^2(\epsilon) = \min\{\delta^1(\frac{\epsilon}{10B}), \delta^1(\tilde{\delta}(\delta'))\} \text{ and } N^2(\epsilon) = \max\{N^1(\frac{\epsilon}{10B}), \tilde{N}(\delta^2(\epsilon))\},$$

where $\tilde{N}$ and $\tilde{\delta}$ are given in (43) and $\delta^1(\epsilon) = \min\{\delta^1_a(\epsilon), \delta^1_b(\epsilon)\}$ and $N^1(\epsilon) = \max\{N^1_a(\epsilon), N^1_b(\epsilon)\}$ with $\delta^1_a(\epsilon), \delta^1_b(\epsilon), N^1_a(\epsilon), N^1_b(\epsilon)$ given in Claim 1.

Suppose that $N > N^2(\epsilon)$ and that $\left|\frac{\lambda^N(Y_k)}{M^N} - \mu_{z^*}(Y_k)\right| < \delta^2(\epsilon)$ for all k. By Claim 1a, because $N > N^2(\epsilon) \ge N^1(\epsilon/10B)$ and for all k, $\left|\frac{\lambda^N(Y_k)}{M^N} - \mu_z(Y_k)\right| < \delta^2(\epsilon) \le \delta^1(\epsilon/10B)$, we have $\tilde{\phi}^N_x[z] > 1 - \epsilon/10B$.

Moreover, because $N > N^1(\tilde{\delta}(\delta'))$ and because $\left|\frac{\lambda^N(Y_k)}{M^N} - \mu_z(Y_k)\right| < \delta^1(\tilde{\delta}(\delta'))$ for all k, it follows from Claim 1a that

$$\tilde{\gamma}^N_z\left[\left|\tilde{\rho}^N(y) - \frac{\mu_z(y)}{\mu_z(Y_k)}\right| < \tilde{\delta}(\delta')\right] > 1 - \tilde{\delta}(\delta') \ge 1 - \frac{\epsilon}{10B}.$$

Now, by (43), it follows that if $\left|\tilde{\rho}^N(y) - \frac{\mu_z(y)}{\mu_z(Y_k)}\right| < \tilde{\delta}(\delta')$, if $\left|\frac{\lambda^N(Y_k)}{M^N} - \mu_z(Y_k)\right| < \tilde{\delta}(\delta')$, and if $N > \tilde{N}(\delta')$, then

$$\max\left\{\left|p^N - \frac{\tilde{R}^N_-}{\tilde{Q}^N_-+b_q}\right|, \left|p^N - \frac{\tilde{Q}^N_-\tilde{R}^N_-}{(\tilde{Q}^N_-+b_q)^2}\right|\right\} < \delta' = \hat{\delta}(\frac{\epsilon}{10B}).$$

This and (42) imply

$$\left| L_z\left(b_q, \frac{\tilde{R}_-^N}{\tilde{Q}_-^N + b_q}, \frac{\tilde{Q}_-^N \tilde{R}_-^N}{(\tilde{Q}_-^N + b_q)^2}\right) - L_z(b_q, p^N, p^N)\right| < \frac{\epsilon}{10B}.$$

Therefore, $\left|\frac{\lambda^N(Y_k)}{M^N} - \mu_z(Y_k)\right| < \tilde{\delta}(\delta')$ and $N > \tilde{N}(\delta')$ imply that

$$\tilde{\gamma}_z^N \left[\left| L_z\left(b_q, \frac{\tilde{R}_-^N}{\tilde{Q}_-^N + b_q}, \frac{\tilde{Q}_-^N \tilde{R}_-^N}{(\tilde{Q}_-^N + b_q)^2}\right) - L_z(b_q, p^N, p^N)\right| < \frac{\epsilon}{10B}\right] > 1 - \frac{\epsilon}{10B}.$$

Combining these results we have

$$\begin{aligned}
& \left|\frac{\partial}{\partial b_q} H_x(b_q, 0; p^N) - \frac{\partial}{\partial b_q} \tilde{H}_x^{\lambda^N}(b_q, 0)\right| \\
\leq\ & \mathbb{E}_{\tilde{\gamma}_z^N}\left[L_z\left(b_q, \frac{\tilde{R}_-^N}{\tilde{Q}_-^N + b_q}, \frac{\tilde{Q}_-^N \tilde{R}_-^N}{(\tilde{Q}_-^N + b_q)^2}\right) - L_z(b_q, p^N, p^N)\right] + (1 - \tilde{\phi}_x^N[z])B \\
<\ & [\frac{\epsilon}{10} + \frac{\epsilon}{10B}(2B)] + [\epsilon/10B]B < \epsilon.
\end{aligned}$$

This establishes (41).

Now we complete the proof of Claim 3 for $\tilde{\beta}^{N,A}$. Let $Q(b_q, b_r; p) = \bar{q} - b_q + \frac{b_r}{p}$. It is straightforward to check that there exists a $D_1 > 0$ such that

$$|b_q - b_q'| + |b_r - b_r'| < D_1 |Q(b;p) - Q(b';p)| \text{ for all } (b, b', p) \in \mathcal{O}^2 \times \left[\frac{\kappa_r}{\bar{q} + \kappa_q}, \frac{\kappa_r + \bar{r}}{\kappa_q}\right].$$

Also, letting

$$M(q;p) = u_q(q, p\bar{q} + \bar{r} - pq; x, z) - u_r(q, p\bar{q} + \bar{r} - pq; x, z)p,$$

we have

$$\frac{\partial}{\partial b_q} H_x(b_q, 0; p^N) = -M(Q(b_q, 0; p^N); p^N) \text{ and } \frac{\partial}{\partial b_r} H_x(0, b_r; p^N) = M(Q(0, b_r; p^N); p^N)/p^N.$$

Now, let D_2 satisfies

$$\begin{aligned}
1/D_2 \ =\ & -\max\{u_{qq}(q, r; x, z) - 2p u_{qr}(q, r; x, z) + p^2 u_{rr}(q, r; x, z) : \\
& (q, r, p) \in \left[d, \bar{q} + \frac{\bar{r}(\bar{q} + \kappa_q)}{\kappa_r}\right] \times \left[d, \bar{r} + \frac{\bar{q}(\bar{r} + \kappa_r)}{\kappa_q}\right] \times \left[\frac{\kappa_r}{\bar{q} + \kappa_q}, \frac{\kappa_r + \bar{r}}{\kappa_q}\right]\},
\end{aligned}$$

where u_{qq}, u_{qr}, and u_{rr} denote second-order derivatives of u. Because u is strictly concave and continuously twice differentiable, D_2 is well-defined and $D_2 > 0$. Moreover,

$$M'(q; p^N) = u_{qq}(q, r; x, z) - 2p^N u_{qr}(q, r; x, z) + (p^N)^2 u_{rr}(q, r; x, z)$$

with $r = p^N \bar{q} + \bar{r} - p^N q$. Hence, $M'(q; p^N) < -1/D_2$ for all $q = Q(b; p^N)$ with $(b, p^N) \in \mathcal{O}^2 \times \left[\frac{\kappa_r}{\bar{q} + \kappa_q}, \frac{\kappa_r + \bar{r}}{\kappa_q}\right]$.

Because the offer $\chi(x; p^N)$ is a "price-taking" offer, it satisfies the first-order conditions at equality, i.e., $M(Q(\chi(x; p^N); p^N); p^N) = 0$. Therefore, by the Mean Value Theorem, for any $\epsilon > 0$, if $|M(\bar{q} - b_q + \frac{b_r}{p^N}; p^N)| < \epsilon/D_1 D_2$ with $b_q b_r = 0$, then

$$|b_q - \chi_q(x; p^N)| + |b_r - \chi_r(x; p^N)| < \epsilon. \tag{44}$$

Let $D = 2D_1 D_2 \frac{\bar{r}+\kappa_r}{\kappa_q}$. Then, for any $p^N \in \left[\frac{\kappa_r}{\bar{q}+\kappa_q}, \frac{\kappa_r+\bar{r}}{\kappa_q}\right]$, $p^N D_1 D_2 < D$.

Now, let $N^3(\epsilon) = N^2(\epsilon/D)$ and $\delta^3(\epsilon) = \delta^2(\epsilon/D)$, where N^2 and δ^2 are given by (41). Suppose that $N > N^2(\epsilon)$ and that $\left|\frac{\lambda^N(Y_k)}{M^N} - \mu_z(Y_k)\right| < \delta^2(\epsilon)$ for all k. We consider three cases.

(a) $\tilde{\beta}_q^{N,A}(x) > 0$.

Then, $\frac{\partial}{\partial b_q} \tilde{H}_x^{\lambda^N}(\tilde{\beta}_q^{N,A}(x), 0) = 0$. By (41), we have $\left|\frac{\partial}{\partial b_q} H_x(\tilde{\beta}_q^{N,A}(x), 0; p^N)\right| < \epsilon/D$ and hence $|M(Q(\tilde{\beta}_q^{N,A}(x), 0; p^N); p^N)| < \epsilon/D$. This, by (44), implies that

$$|\tilde{\beta}_q^{N,A}(x) - \chi_q(x; p^N)| + |\chi_r(x; p^N)| < \epsilon.$$

(b) $\tilde{\beta}_r^{N,A}(x) > 0$.

Then, $\frac{\partial}{\partial b_r} \tilde{H}_x^{\lambda^N}(0, \tilde{\beta}_r^{N,A}(x)) = 0$. By (41), we have $\left|\frac{\partial}{\partial b_r} H_x(0, \tilde{\beta}_r^{N,A}(x); p^N)\right| < \epsilon/D$ and hence $|M(Q(\tilde{\beta}_q^{N,A}(x), 0; p^N))/p^N| < \epsilon/D$. This, (44), and $D/p^N > D_1 D_2$ for all $p^N \in \left[\frac{\kappa_r}{\bar{q}+\kappa_q}, \frac{\kappa_r+\bar{r}}{\kappa_q}\right]$ imply

$$|\chi_q(x; p^N)| + |\tilde{\beta}_r^{N,A}(x) - \chi_r(x; p^N)| < \epsilon.$$

(c) $\tilde{\beta}_q^{N,A}(x) = 0 = \tilde{\beta}_r^{N,A}(x)$.

Then, $\frac{\partial}{\partial b_q} \tilde{H}_x^{\lambda^N}(0,0) \leq 0$ and $\frac{\partial}{\partial b_r} \tilde{H}_x^{\lambda^N}(0,0) \leq 0$. By (41), we have

$$-M(Q(0,0; p^N); p^N) = \frac{\partial}{\partial b_q} H_x(0,0; p^N) < \frac{\partial}{\partial b_q} H_x^{\lambda^N}(0,0) + \epsilon/D \leq \epsilon/D$$

and

$$M(Q(0,0; p^N); p^N)/p^N = \frac{\partial}{\partial b_r} H_x(0,0; p^N) < \frac{\partial}{\partial b_r} H_x^{\lambda^N}(0,0) + \epsilon/D \leq \epsilon/D.$$

It then follows that $|M(Q(0,0; p^N); p^N)| < \epsilon/D_1 D_2$ and hence

$$|\chi_q(x; p^N)| + |\chi_r(x; p^N)| < \epsilon.$$

This concludes the proof of Claim 3 for $\tilde{\beta}^{N,A}$.

The argument is identical for $\hat{\beta}^{N,A}$, except that we need an additional argument to show that for any $\epsilon > 0$, there exists $N^2(\epsilon)$ and $\delta^2(\epsilon)$ such that if $N > N^2(\epsilon)$ and if $\left|\frac{\lambda^N(Y_k)}{M^N} - \mu_z(Y_k)\right| < \delta^2(\epsilon)$ for all k, then for all $b_q \in [0, \bar{q} - d]$ and all $b_r \in [0, \bar{r} - d]$,

$$\left|\frac{\partial}{\partial b_q} H_x(b_q, 0; p^N) - \frac{\partial}{\partial b_q} \hat{H}_x^{\lambda^N}(b_q, 0)\right| < \epsilon \text{ and } \left|\frac{\partial}{\partial b_r} H_x(0, b_r; p^N) - \frac{\partial}{\partial b_r} \hat{H}_x^{\lambda^N}(0, b_r)\right| < \epsilon. \tag{45}$$

Although (45) is completely analogous to (41), an additional argument is required because $\tilde{\beta}^{N,A}$ appears in $(\hat{Q}_-^N, \hat{R}_-^N)$, while p^N only involves $\hat{\beta}^{N,A}$.

Notice that

$$p^N - \frac{\hat{R}_-^N}{\hat{Q}_-^N} = \frac{\sum_{y\in X} \mu_z(y)\hat{\beta}_r^{N,A}(y) + \kappa_r}{\sum_{y\in X} \mu_z(y)\hat{\beta}_q^{N,A}(y) + \kappa_q} - \frac{\sum_{k=1,..,K} \sum_{y\in Y_k} \frac{\lambda_-^N(Y_k)}{M^N}\hat{\rho}^N(y)\hat{\beta}_r^{N,A}(y) + \frac{1}{M^N}\tilde{\beta}_r^{N,A}(\tilde{\xi}) + \kappa_r}{\sum_{k=1,..,K} \sum_{y\in Y_k} \frac{\lambda_-^N(Y_k)}{M^N}\hat{\rho}^N(y)\hat{\beta}_q^{N,A}(y) + \frac{1}{M^N}\tilde{\beta}_q^{N,A}(\tilde{\xi}) + \kappa_q}.$$

Therefore,

$$\left|p^N - \frac{\hat{R}_-^N}{\hat{Q}_-^N}\right| \leq \frac{2(\bar{q}+\kappa_q)(\bar{r}+\kappa_r)}{\kappa_q^2}\left[\sum_{k=1}^{K}\left|\mu_z(Y_k) - \frac{\lambda_-^N(Y_k)}{M^N}\hat{\rho}^N(y)\right| + \left|\frac{\bar{q}}{M^N} + \frac{\bar{r}}{M^N}\right|\right].$$

Also,

$$\left|\frac{\hat{Q}_-^N\hat{R}_-^N}{(\hat{Q}_-^N+b_q)^2} - \frac{\hat{R}_-^N}{\hat{Q}_-^N}\right| \leq \frac{\bar{r}+\kappa_r}{\kappa_q}\frac{\bar{q}(2\kappa_q + \frac{\bar{q}}{M^N})}{\kappa_q^2 M^N} \text{ and } \left|\frac{\hat{R}_-^N}{\hat{Q}_-^N+b_q} - \frac{\hat{R}_-^N}{\hat{Q}_-^N}\right| \leq \frac{\bar{r}+\kappa_r}{\kappa_q}\frac{\bar{q}}{\kappa_q M^N}.$$

Thus, for any $\epsilon > 0$ there exist $\tilde{N}(\epsilon)$ and $\tilde{\delta}(\epsilon) \leq \epsilon$ such that if $N > \tilde{N}(\epsilon)$ and if $\left|\hat{\rho}^N(y) - \frac{\mu_z(y)}{\mu_z(Y_k)}\right| < \tilde{\delta}(\epsilon)$ and $\left|\frac{\lambda^N(Y_k)}{M^N} - \mu_z(Y_k)\right| < \tilde{\delta}(\epsilon)$, then

$$\left|p^N - \frac{\hat{R}_-^N}{\hat{Q}_-^N}\right| < \epsilon,\ \left|\frac{\hat{Q}_-^N\hat{R}_-^N}{(\hat{Q}_-^N+b_q)^2} - p^N\right| < \epsilon,\ \text{and}\ \left|\frac{\hat{R}_-^N}{\hat{Q}_-^N+b_q} - p^N\right| < \epsilon$$

for all b_q. Given these results, the rest of the argument is exactly the same as for $\tilde{\beta}^{N,A}$.□

Claim 4. For any $\epsilon > 0$, there exists $N^4(\epsilon)$ such that if $N > N^4(\epsilon)$, then for all $x \in X$,

$$|F_x^N(A) - F_x^N(s_1^N(x))| < \epsilon \text{ for any } A \subseteq X, \tag{46}$$

where, recall, $F_x^N(A)$ is the expected payoff from offer a that is associated with the set A at stage-1 conditional on being active.

Proof. Consider a state z. First we show that for any $\epsilon > 0$, there exist $N^5(\epsilon)$ and $\delta^5(\epsilon)$ such that if $N \geq N^5(\epsilon)$ and if $\left|\lambda^N(Y_k)/M^N - \mu_z(Y_k)\right| < \delta^5(\epsilon)$, then for each $x \in X$,

$$\| \hat{\beta}^{N,A}(x) - \beta^{z,\kappa}(x) \| < \epsilon \text{ and } \| \tilde{\beta}^{N,A}(x) - \beta^{z,\kappa}(x) \| < \epsilon, \tag{47}$$

where $\| b - b' \| = |b_q - b'_q| + |b_r - b'_r|$ for all $b, b' \in \mathcal{O}$, and recall that $\beta^{z,\kappa}$ is the offer corresponding to the CE in $\mathcal{L}^z(\kappa)$. We establish (47) for $\hat{\beta}^{N,A}$ and a fixed state z^*. The other case is exactly the same.

If we set

$$\tilde{\delta} = (\tilde{\delta}_q, \tilde{\delta}_r) = \sum_{x\in X} \mu_{z^*}(x)[\beta^N(x) - \chi(x; p^N)],$$

then, by construction, $(\chi(x;p^N))_{x\in X}$ satisfies

$$p^N = \frac{\sum_{x\in X}\mu_z(x)\chi_r(x;p^N) + \kappa_r + \tilde{\delta}_r}{\sum_{x\in X}\mu_z(x)\chi_q(x;p^N) + \kappa_q + \tilde{\delta}_q}.$$

That is, $(\chi(x;p^N))_{x\in X}$ is an offer corresponding to the unique CE in $\mathcal{L}^z(\kappa+\tilde{\delta})$. Moreover, the CE's varies continuously with $\tilde{\delta}$. Thus, there exists a $\delta^P(\epsilon) \le \epsilon$ such that if $\max\{|\tilde{\delta}_q|, |\tilde{\delta}_r|\} \le \delta^P(\epsilon)$, then

$$\|\beta^{z,\kappa}(x) - \chi(x;p^N)\| < \epsilon \text{ for all } x \in X. \tag{48}$$

Now for any $\epsilon > 0$, let $\delta' = \delta^P(\epsilon/2)$ and let $\delta^5(\epsilon) = \delta^3(\delta')$. Let $N^5(\epsilon) = N^3(\delta')$. By Claim 3, if $\left|\frac{\lambda^N(Y_k)}{M^N} - \mu_z(Y_k)\right| < \delta^3(\delta')$ for all k and if $N > N^3(\delta')$, then

$$\|\tilde{\beta}^{N,A}(x) - \chi(x;p^N)\| < \delta' \le \frac{\epsilon}{2} \text{ for all } x \in X.$$

This then implies that

$$|\tilde{\delta}_q|+|\tilde{\delta}_r| \le \sum_{x\in X}\mu_{z^*}(x)|\tilde{\beta}^{N,A}_q(x)-\chi_q(x;p^N)|+\sum_{x\in X}\mu_z(x)|\tilde{\beta}^{N,A}_r(x)-\chi_r(x;p^N)| < \delta' = \delta^P(\epsilon/2).$$

By (48), this implies that $\|\beta^{z,\kappa}(x) - \chi(x;p)\| < \epsilon/2$. Thus, for each x,

$$\|\tilde{\beta}^{N,A}(x) - \beta^{z,\kappa}(x)\| \le \|\tilde{\beta}^{N,A}(x) - \chi(x;p^N)\| + \|\beta^{z,\kappa}(x) - \chi(x;p^N)\| < \epsilon,$$

which is (47).

Let $\bar{\gamma}^N_z$ be the probability distribution over other active agents' types conditional on state z, that is, $\bar{\gamma}^N_z[\xi_1, ..., \xi_{M^N-1}] = \prod_{t=1}^{M^N}\mu_z(\xi_t)$. For any nonempty $A \subseteq X$,

$$F^N_x(A) = \sum_{z\in Z}\tau_x[z]\mathbb{E}_{\bar{\gamma}^N_z}\left[u\left(\bar{q} + \frac{\tilde{\beta}^{N,A}_r(x)Q^N_- - \tilde{\beta}^{N,A}_q(x)R^N_-}{\tilde{\beta}^{N,A}_r(x) + R^N_-}, \bar{r} + \frac{\tilde{\beta}^{N,A}_q(x)R^N_- - \tilde{\beta}^{N,A}_r(x)Q^N_-}{\tilde{\beta}^{N,A}_q(x) + Q^N_-}; x, z\right)\right],$$

where

$$(Q^N_-, R^N_-) = \sum_{y\in X}\sigma(y)\hat{\beta}^{N,A}(y) + M^N\kappa \text{ and } \sigma(y) = \#\{\xi_t : t = 1, ..., M^N-1, \xi_t = y\}.$$

By the *law of large numbers*, $\sigma(y)/M^N$ converges to $\mu_z(y)$ in probability under $\bar{\gamma}^N_z$. As a result, $\lambda^N(Y_k)/M^N$ converges to $\mu_z(Y_k)$ in probability under $\bar{\gamma}^N_z$. Therefore, by (47), for any $\epsilon' > 0$ there exists $N^z(\epsilon')$ such that if $N > N^z(\epsilon')$, then

$$\bar{\gamma}^N_z\left[\left|\left(\frac{\tilde{\beta}^{N,A}_r(x)Q^N_- - \tilde{\beta}^{N,A}_q(x)R^N_-}{\tilde{\beta}^{N,A}_r(x) + R^N_-}\right) - \left(\frac{\beta^{z,\kappa}_r(x)}{p^{z,\kappa}} - \beta^{z,\kappa}_q(x)\right)\right| < \epsilon'\right] > 1-\epsilon'$$

and

$$\bar{\gamma}^N_z\left[\left|\left(\frac{\tilde{\beta}^{N,A}_q(x)R^N_- - \tilde{\beta}^{N,A}_r(x)Q^N_-}{\tilde{\beta}^{N,A}_q(x) + Q^N_-}\right) - \left(\beta^{z,\kappa}_q(x)p^{z,\kappa} - \beta^{z,\kappa}_r(x)\right)\right| < \epsilon'\right] > 1-\epsilon'.$$

With appropriate ϵ''s, this implies that

$$\left| F_x^N(A) - \sum_{z\in Z} \tau_x[z] u\left(\bar{q} + \frac{\beta_r^{z,\kappa}(x)}{p^{z,\kappa}} - \beta_q^{z,\kappa}(x), \bar{r} + \beta_q^{z,\kappa}(x) p^{z,\kappa} - \beta_r^{z,\kappa}(x); x, z\right)\right| < \epsilon.$$

Claim 4 follows from the fact that there are only finitely many nonempty $A \subseteq X$. $\square$

Now we complete the proof. Recall that we begin with a candidate semi-pooling equilibrium associated with the partition $\mathcal{Y} = \{Y_1, ..., Y_K\}$ with $1 < K < |X|$. Because such an equilibrium does not exist when $|X| = 2$, we may assume that for some $y^1, y^2 \in X$, $y^1 \neq y^2 \in Y_1$. Recall that $G_x(a)$ is the objective function for a type-x agent at stage-1 conditional on being inactive (see (6)). Because $y^1 \neq y^2$, there exists $C > 0$ such that for any $a \in \mathcal{O}$, either $G_{y^1}(a) < G_{y^1}(\alpha^*(y^1)) - C$ or $G_{y^2}(a) < G_{y^2}(\alpha^*(y^1)) - C$. Assume without loss of generality that $G_{y^1}(s_1^N(Y_1)) < G_{y^1}(\alpha^*(y^1)) - C$ so that $s_1^N(y^1) \neq \alpha^*(y^1)$. Consider then a potential defector of type y_1.

Let $\bar{N} = N^4(\frac{\eta C}{2(1-\eta)})$. Then, by Claim 4, if $N \geq \bar{N}$,

$$|F_{y^1}^N(s_1^N(Y_1)) - F_{y^1}^N(\alpha^*(y^1))| < \frac{\eta C}{2(1-\eta)} \text{ and } |F_{y^2}^N(s_1^N(Y_1)) - F_{y^2}^N(\alpha^*(y^2))| < \frac{\eta C}{2(1-\eta)}.$$

Then, for $N \geq \bar{N}$,

$$\begin{aligned} & \eta G_{y^1}(s_1^N(Y_1)) + (1-\eta) F_{y^1}^N(s_1^N(Y_1)) \\ < \;& \eta(G_{y^1}(\alpha^*(y^1)) - C) + (1-\eta)(F_{y^1}^N(\alpha^*(y^1)) + \frac{\eta C}{2(1-\eta)}) \\ = \;& \eta G_{y^1}(\alpha^*(y^1)) + (1-\eta) F_{y^1}^N(\alpha^*(y^1)) - \frac{\eta C}{2}. \end{aligned}$$

Hence, deviating from $s_1^N(y^1)$ to $\alpha^*(y^1)$ is profitable, a contradiction. This shows that s^N is a separating equilibrium. ■

Corollary 2. Suppose that the conditions in Theorem 2 hold. For any sequence of equilibria $\{(s_1^N, s_2^N)\}_{N=1}^{\infty}$, if $(q_x^N, r_x^N)_{x\in X}$ is the stage-2 consumption for agents of type-x in (s_1^N, s_2^N), then, for each $z \in Z$,

$$\lim_{N\to\infty} (q_x^N, r_x^N)_{x\in X} = (q_x^z, r_x^z)_{x\in X}$$

almost surely conditional on z, where $(q_x^z, r_x^z)_{x\in X}$ is the competitive allocation in $\mathcal{L}^z(\kappa)$ for each $z \in Z$.

Proof. Notice that by Theorem 2, there exists $\bar{N}$ (independent of the choice of equilibria) such that $N > \bar{N}$ implies that any equilibrium of our mechanism in the N-agent economy is separating. Thus, stage-2 behavior is described by a symmetric equilibrium β^σ of the game $\mathcal{E}(M^N, \sigma^N/M^N, \phi^{\sigma^N}, \kappa)$ for a type configuration σ^N with M^N active agents, as described in Proposition 1. This proof applies the Theorem of the Maximum to a sequence of such equilibria with σ^N/M^N converges to the limit distribution μ_z. We can write the best-response objective (see (20)) as

$$H_x(b; Q_-, R_-, \phi) = \sum_{z'\in Z} \phi(z') u\left(q, r; x, z'\right), \tag{49}$$

with

$$q=\bar{q}+\frac{b_r}{p(1+\frac{b_r}{R_-})}-\frac{b_q}{1+\frac{b_r}{R_-}},\ \text{and}\ r=\bar{r}-\frac{b_r}{1+\frac{b_q}{Q_-}}+\frac{pb_q}{1+\frac{b_q}{Q_-}},$$

and $p=R_-/Q_-$.

Now, let

$$F_x(b;p,c_1,c_2,\phi)=\sum_{z'\in Z}\phi(z')u\left(q,r;x,z'\right)$$

with

$$q=\bar{q}+\frac{b_r}{p(1+c_2b_r)}-\frac{b_q}{1+c_2b_r},\ \text{and}\ r=\bar{r}-\frac{b_r}{1+c_1b_q}+\frac{pb_q}{1+c_1b_q},$$

and where the domain for F_x is $A=\mathcal{O}\times\left[\frac{\kappa_r}{\bar{q}+\kappa_q},\frac{\bar{r}+\kappa_r}{\kappa_q}\right]\times\left[0,\frac{1}{\kappa_r}\right]\times\left[0,\frac{1}{\kappa_q}\right]\times\Delta(Z)$. It follows that $F_x(b;p,1/Q_-,1/R_-,\phi)=H_x(b;Q_-,R_-,\phi)$. Therefore, by the argument used in the proof of proposition 1, $F_x(\cdot;p,c_1,c_2,\phi)$ has a unique maximum, $g_x(p,c_1,c_2,\phi)$. And because F_x is continuous on A, the Theorem of the Maximum implies that $g_x(p,c_1,c_2,\phi)$ is continuous.

Now consider

$$H_x(b;Q_-^N,R_-^N,\phi^N)=F_x(b;\frac{R_-^N}{Q_-^N},\frac{1}{Q_-^N},\frac{1}{R_-^N},\phi^N)$$

with

$$(Q_-^N,R_-^N)=\sum_{y\in X}\sigma^N(y)(\beta^N(y)+\kappa)-\beta^N(x)$$

and with ϕ^N being the common prior derived from σ^N using Bayes' rule. Notice that $(\frac{R_-^N}{Q_-^N},\frac{1}{Q_-^N},\frac{1}{R_-^N})\in\left[\frac{\kappa_r}{\bar{q}+\kappa_q},\frac{\bar{r}+\kappa_r}{\kappa_q}\right]\times\left[0,\frac{1}{\kappa_r}\right]\times\left[0,\frac{1}{\kappa_q}\right]$. Therefore, by the definition of β^N, $\beta^N(x)=g_x(\frac{R_-^N}{Q_-^N},\frac{1}{Q_-^N},\frac{1}{R_-^N},\phi)$. Because $\{\beta^N\}_{N=1}^{\infty}$ is bounded, it has a convergent subsequence, say $\{\beta^{N_s}\}_{s=1}^{\infty}$, with limit denoted $\hat{\beta}$. Notice that $\lim_{N\to\infty}\phi^N=\delta_z$, where $\delta_z[z']=1$ for $z'=z$ (and 0 otherwise). By the continuity of g_x, it follows that

$$\begin{aligned}\hat{\beta}(x) &= \lim_{s\to\infty}g_x(\frac{R_-^{N_s}}{Q_-^{N_s}},\frac{1}{Q_-^{N_s}},\frac{1}{R_-^{N_s}},\phi^{N_s})\\ &= g_x(\lim_{s\to\infty}\frac{R_-^{N_s}}{Q_-^{N_s}},\lim_{s\to\infty}\frac{1}{Q_-^{N_s}},\lim_{s\to\infty}\frac{1}{R_-^{N_s}},\lim_{s\to\infty}\phi^{N_s})=g_x(\hat{p},0,0,\delta_z),\end{aligned}$$

where

$$\hat{p}=\frac{\sum\mu_z(y)\hat{\beta}_r(y)+\kappa_r}{\sum\mu_z(y)\hat{\beta}_q(y)+\kappa_q}.$$

By the definition of F_x, it follows that $\hat{\beta}(x)$ maximizes $u(\bar{q}-b_q+\frac{b_r}{\hat{p}},\bar{r}-b_r+\hat{p}b_q;x,z)$. Therefore, it is an offer corresponding to the CE in $\mathcal{L}^z(\kappa)$. It follows that $\hat{\beta}=\beta^{z,\kappa}$. Because this convergence holds for all convergent subsequences and because the sequence $\{\beta^{\sigma^N}\}$ is bounded, we have $\lim_{N\to\infty}\beta^{\sigma^N}=\beta^{z,\kappa}$.

Finally, because $\lim_{N\to\infty}\sigma^N(x)/\lceil(1-\eta)N\rceil=\mu_z(x)$ for each $x\in X$ holds almost surely conditional on z, the convergence result follows from the above arguments. ■

7.3 Optimality

Theorem 3. Suppose that $\alpha^*(x) \neq \alpha^*(y)$ for any $x \neq y$. Let $\varepsilon > 0$ be given.

(i) Suppose that for each $z \in Z$, $\mathcal{L}^z$ has a regular competitive equilibrium where every type trades. There exists $\bar{\kappa}$ and $N(\kappa, \eta)$ such that if (κ, η) satisfies

$$\bar{\kappa} > \kappa \geq 4(\bar{q}, \bar{r})\eta > 0 \tag{50}$$

and $N > N(\kappa, \eta)$, then there exists a separating equilibrium whose outcome is ex post $(\varepsilon, 3\kappa/2)$-efficient.

(ii) Suppose that the competitive demand for good-q is monotone in $\mathcal{L}^z$ for all $z \in Z$, and suppose that A1-A3 hold. There exists $\bar{\kappa}$ and $N(\kappa, \eta)$ such that if (κ, η) satisfies (50) and $N > N(\kappa, \eta)$, then the outcome of any symmetric equilibrium in pure strategies is ex post $(\varepsilon, 3\kappa/2)$-efficient.

Proof of (i). Let $\{(q_x^{z,\kappa}, r_x^{z,\kappa})\}_{x\in X}$ be a competitive allocation in the economy $\mathcal{L}^z(\kappa)$ (see Proposition 2 (ii)) which is regular and in which every type trades. Consider another economy $\mathcal{J}^z(\rho; \kappa)$, where $\rho \in \Delta(X)$ is the proportion of agents according to type, and each agent has endowment $(\bar{q} + \kappa_q, \bar{r} + \kappa_r)$. Let $\{(q_x^{z,\rho,\kappa}, r_x^{z,\rho,\kappa})\}_{x\in X}$ denote the competitive allocation for $\mathcal{J}^z(\rho; \kappa)$ under known state-of-the-world z. We omit the proof of the following claim, which only asserts continuity of competitive allocations w.r.t. endowment parameters and the proportion of different types.

Claim 1. Let $\{(q_x^z, r_x^z)\}_{x\in X}$ be a regular competitive allocation in $\mathcal{L}^z$, and, let $(q_x^{z,\kappa}, r_x^{z,\kappa})_{x\in X}$ be the competitive allocation in $\mathcal{L}^z(\kappa)$ that is close to $\{(q_x^z, r_x^z)\}_{x\in X}$ as constructed in Proposition 2 (ii). Then, for any $\epsilon > 0$, there exists $\delta^1(\epsilon) > 0$ such that if $|\rho(x) - \mu_z(x)| < \delta^1(\epsilon)$ for each $x \in X$ and if $\max\{\kappa_q, \kappa_r\} < \delta^1(\epsilon)$, then there is a competitive allocation $\{(q_x^{z,\rho,2\kappa}, r_x^{z,\rho,2\kappa})\}_{x\in X}$ in $\mathcal{J}^z(\rho; 2\kappa)$ for which

$$\left| u(q_x^{z,\kappa}, r_x^{z,\kappa}; x, z) - u(q_x^{z,\rho,2\kappa}, r_x^{z,\rho,2\kappa}; x, z) \right| < \epsilon.$$

for all $x \in X$.

The next claim constructs the high probability event $E_{z,c^N}(\epsilon)$ that we need to establish (ε, δ) ex post optimality of a Theorem-1 equilibrium. The event E_{z,c^N} will be the intersection of two events, E^1_{z,c^N} and E^2_{z,c^N}, where the first involves only exogenous random variables and the second depends on a selected equilibrium.

Fix some $(\kappa, \eta) > 0$ (recall that η is the probability of being inactive) such that $\mathcal{L}^z(\kappa)$, for each $z \in Z$, has a regular competitive equilibrium allocation, denoted by $(q_x^{z,\kappa}, r_x^{z,\kappa})_{x\in X}$, where every type trades. For any realization ζ^N and c^N, there is a unique corresponding type-configuration for active agents, denoted $\sigma(\zeta^N, c^N) = (\sigma(\zeta^N, c^N)(x) : x \in X)$. For each $(z, c^N) \in Z \times \mathbb{C}^N$ and for any $\epsilon > 0$, define the event $E^1_{z,c^N}(\epsilon)$ as

$$E^1_{z,c^N}(\epsilon) = \left\{ \zeta^N : (\forall x) \left| \sigma(\zeta^N, c^N)(x)/M^N - \mu_z(x) \right| < \delta^1(\epsilon) \right\}, \tag{51}$$

where $\delta^1(\epsilon)$ is defined in Claim 1 above (uniformly across all z's). By Theorem 1 (as well as Corollary 1), there exists $\bar{N}(\kappa, \eta)$ such that if $N > \bar{N}(\kappa, \eta)$, then there exists a separating equilibrium (s_1^N, s_2^N) (corresponding to the competitive allocations $\{(q_x^{z,\kappa}, r_x^{z,\kappa})_{x\in X} : z \in Z\}$). As above, we use β^σ to denote the stage-2 offers along the corresponding equilibrium path for a realization of type-configuration σ for active agents; we also use $(q^\sigma(x), r^\sigma(x))$ to denote the corresponding payoffs as determined in (3) from offers β^σ. Then, let

$$E^2_{z,c^N}(\epsilon) = \left\{\zeta^N : (\forall x) \left| u(q^{\sigma(\zeta^N,c^N)}(x), r^{\sigma(\zeta^N,c^N)}(x); x, z) - u(q_x^{z,\kappa}, r_x^{z,\kappa}; x, z)\right| < \epsilon\right\}. \quad (52)$$

Claim 2. Let $E_{z,c^N}(\epsilon) = E^1_{z,c^N}(\epsilon) \cap E^2_{z,c^N}(\epsilon)$. There exists $N^2(\kappa, \eta, \epsilon)$ such that if $N > N^2(\kappa, \eta, \epsilon)$, then for any $(z, c^N) \in Z \times \mathbb{C}^N$, $\mathbb{P}\left[E_{z,c^N}(\epsilon) \mid c^N, z\right] > 1 - \epsilon$.

Proof. Let ξ be an infinite sequence of i.i.d. X-valued random variables with marginal distribution given by μ_z. Because c^N is independent of the realization of types and the state-of-the-world, for any N the sequence $\{\zeta_n : 1 \leq n \leq N,\ c_n = 1\}$ and the sequence $\{\xi_m : m = 1, ..., M^N\}$ have the same distribution conditional on z and c^N. For each N and $\xi^{M^N} = (\xi_1, ..., \xi_{M^N})$, let β^{σ^N} describe the equilibrium stage-2 offers under (s_1^N, s_2^N) along the equilibrium path with $\sigma^N(x) = \#\{1 \leq m \leq M^N : \xi_m = x\}$ and let $(q^{\sigma^N}, r^{\sigma^N})$ describe the corresponding equilibrium payoffs for active agents. By the law of large numbers, for any z and $x \in X$,

$$\lim_{N\to\infty} \sigma^N(x)/M^N = \mu_z(x)$$

almost surely conditional on z. Thus, by the construction in Theorem 1 and by Proposition 2, it follows that $\lim_{N\to\infty} \beta^{\sigma^N} = \beta^z$ almost surely conditional on z. Hence, by continuity of u, for any z,

$$\lim_{N\to\infty} u\left(q^{\sigma^N}(x), r^{\sigma^N}(x); x, z\right) = u(q_x^{z,\kappa}, r_x^{z,\kappa}; x, z)$$

almost surely conditional on z and hence it converges in probability conditional on z. ■

Now we can complete the proof. Given ε, let $\bar{\kappa} = \delta^1(\frac{\varepsilon}{3})$ and consider our mechanism with $\max\{\kappa_q, \kappa_r\} < \bar{\kappa}$ and with $\eta \leq \min\{\frac{\kappa_q}{4\bar{q}}, \frac{\kappa_r}{4\bar{r}}\}$.

Given the separating equilibrium (s_1^N, s_2^N) constructed above for $N > \bar{N}(\kappa, \eta)$ agents, and given a type realization, $\zeta^N = (\zeta_1, ..., \zeta_N) \in X^N$, and an activeness-status realization, $c^N = (c_1, ..., c_N)$, the corresponding allocation is as follows: if $c_n = 0$, then

$$\omega_n^{s^N}(\zeta^N, c^N, z) = \left(\bar{q} - \alpha_q(\zeta_n) + \frac{\alpha_r(\zeta_n)}{p^1}, \bar{r} - \alpha_r(\zeta_n) + \alpha_q(\zeta_n)p^1\right),$$

where $s_1^N(x) = (\alpha_q(x), \alpha_r(x))$ for all $x \in X$; if $c_n = 1$, then

$$\omega_n^{s^N}(\zeta^N, c^N, z) = (q^{\sigma(\zeta^N,c^N)}(\zeta_n), r^{\sigma(\zeta^N,c^N)}(\zeta_n)).$$

Let $N(\kappa, \eta) = N^2(\kappa, \eta, \frac{\varepsilon}{3})$. Now we show that if $\max\{\kappa_q, \kappa_r\} < \bar{\kappa}$ and $\eta \leq \min\{\frac{\kappa_q}{4\bar{q}}, \frac{\kappa_r}{4\bar{r}}\}$, and if $N > N(\kappa, \eta)$, then the allocation $\{\omega_n^{s^N} : n \in \mathcal{N}\}$ is ex post $(\varepsilon, 1.5\kappa)$-efficient. For any realization $\zeta^N = (\zeta_1, ..., \zeta_N) \in X^N$ and $c^N = (c_1, ..., c^N)$, let $\mathcal{M}(\zeta^N, c^N) = \{m \in \mathcal{N} : c_m = 1\}$.

Then,

$$\begin{aligned}
\sum_{n\in\mathcal{N}} \omega_n^{s^N}(\zeta^N, c^N, z) &= \sum_{m\in\mathcal{M}(\zeta^N,c^N)} \omega_m^{s^N}(\zeta^N, c^N, z) + \sum_{n\in\mathcal{N}-\mathcal{M}(\zeta^N,c^N)} \omega_n^{s^N}(\zeta^N, c^N, z) \\
&\leq (1-\eta)N(\bar{q}+\kappa_q, \bar{r}+\kappa_r) + \eta N(2\bar{q}, 2\bar{r}) \\
&\leq N[(\bar{q},\bar{r}) + (\kappa_q,\kappa_r) + (\frac{\kappa_q}{2}, \frac{\kappa_r}{2})] = N[(\bar{q},\bar{r}) + 1.5(\kappa_q,\kappa_r).
\end{aligned}$$

Therefore, ω^{s^N} is 1.5κ-feasible.

For each z and for each c^N, by claim 2, $N > N^2(\kappa, \eta, \frac{\varepsilon}{3})$ implies $\mathbb{P}[E_{z,c^N}(\varepsilon/3)|z, c^N] > 1-\varepsilon$. Thus, to show that ω^{s^N} is ex post $(\varepsilon, 1.5\kappa)$-efficient, suppose, by way of contradiction, that ω' is 1.5κ-feasible and (15) holds for some z, c^N and for some $\zeta^N \in E_{z,c^N}(\varepsilon/3)$. Fix such a (z, c^N, ζ^N) and let $\omega'_n(\zeta^N, c^N, z) = (q'_n, r'_n)$ for each $n \in \mathcal{N}$.

Let $\rho(x) = \sigma(\zeta^N, c^N)(x)/M^N$. Then, let $(q^{z,2\kappa,\rho}, r^{z,2\kappa,\rho_x})_{x\in X}$ be the competitive allocation for a finite economy which has $\sigma(\zeta^N, c^N)(x)$ agents of type-x for each x and in which each agent has endowment $(\bar{q}+2\kappa_q, \bar{r}+2\kappa_r)$. Now, consider the following allocation $\{(q''_n, r''_n) : n \in \mathcal{N}\}$: $(q''_n, r''_n) = (q_x^{z,\rho,2\kappa}, r_x^{z,\rho,2\kappa})$, as constructed in Claim 1, if $c_n = 1$ and $\zeta_n = x$; and $(q''_n, r''_n) = (0,0)$ if $c_n = 0$. Because $\zeta^N \in E^1_{z,c^N}(\varepsilon/3)$ and $\max\{\kappa_q, \kappa_r\} < \delta^1(\varepsilon/3)$, it follows from Claim 1 that

$$\left|u(q_x^{z,\kappa}, r_x^{z,\kappa}; x, z) - u(q_x^{z,\rho,2\kappa}, r_x^{z,\rho,2\kappa}; x, z)\right| < \varepsilon/3.$$

Moreover, because $\zeta^N \in E^2_{z,c^N}(\varepsilon/3)$, we have

$$\left|u(q^{\sigma(\zeta^N,c^N)}(x), r^{\sigma(\zeta^N,c^N)}(x); x, z) - u(q_x^{z,\kappa}, r_x^{z,\kappa}; x, z)\right| < \varepsilon/3.$$

Thus,

$$u(q_x^{z,\rho,2\kappa}, r_x^{z,\rho,2\kappa}; x, z) < u(q^{\sigma(\zeta^N,c^N)}(x), r^{\sigma(\zeta^N,c^N)}(x); x, z) + 2\varepsilon/3.$$

Now, for each n such that $c_n = 1$ and $\zeta_n = x$,

$$u(q''_n, r''_n; x, z) < u(\omega_n^{s^N}(\zeta^N, c^N, z); x, z) + 2\varepsilon/3 < u(q'_n, r'_n; x, z) - \varepsilon/3; \tag{53}$$

while for each n such that $c_n = 0$ and $\zeta_n = x$,

$$u(q''_n, r''_n; x, z) = u(0, 0; x, z) \leq u(\omega_n^{s^N}(\zeta^N, c^N, z); x, z) < u(q'_n, r'_n; x, z) - \varepsilon, \tag{54}$$

where the second inequality in each of (53) and (54) follows from (15), the contradicting assumption. Therefore, $\{(q'_n, r'_n) : n \in \mathcal{N}\}$ Pareto dominates $\{(q''_n, r''_n) : n \in \mathcal{N}\}$.

However, $\{(q''_n, r''_n) : n \in \mathcal{N}\}$ is a competitive allocation (with inactive agents having zero endowments) for an economy with total resources no less than that for the allocation $\{(q'_n, r'_n) : n \in \mathcal{N}\}$. By the first fundamental theorem of welfare economics, it follows that $\{(q''_n, r''_n) : n \in \mathcal{N}\}$ cannot be Pareto dominated by $\{(q'_n, r'_n) : n \in \mathcal{N}\}$.

Proof of (ii). Under A1-A3 and by Theorems 1 and 2, it follows that there is a number $\tilde{N}(\kappa, \eta)$ such that if $N > \tilde{N}(\kappa, \eta)$, then any symmetric equilibrium in pure strategies

(s_1^N, s_2^N) is a separating equilibrium. For any $\varepsilon > 0$ we can then construct $\bar{\kappa}$ and $N(\kappa, \eta)$ as in the proof of (i) to show ex post $(\varepsilon, 1.5\kappa)$-efficiency. However, we need to modify Claim 2 in that proof so that the convergence rate does not depend on the sequence of equilibria that we choose. Claim 3 is the modified version. There, as in Claim 2, δ^1 comes from Claim 1.

Claim 3. Let (κ, η) be given. Let $N > \tilde{N}(\kappa, \eta)$. Then, for any $\epsilon > 0$, there exists $N^3(\kappa, \eta, \epsilon) > \tilde{N}(\kappa, \eta)$ such that if $N > N^3(\kappa, \eta, \epsilon)$, then for any equilibrium (s_1^N, s_2^N) and for any $(z, c^N) \in Z \times \mathbb{C}^N$, the event $E_{z,c^N}(\epsilon)$ given by (51) and (52) w.r.t. the equilibrium (s_1^N, s_2^N) satisfies $\mathbb{P}\left[E_{z,c^N}(\epsilon) \mid c^N, z\right] > 1 - \epsilon$.

Proof. In the proof of Theorem 2, we establish that for any semi-pooling equilibrium (including the separating equilibrium), the convergence rate of β^{σ^N} to β^z (as defined in Proposition 2) is uniformly bounded across all equilibria. When $N > \tilde{N}(\kappa, \eta)$, all such equilibria are separating. Indeed, by equation (47) (taking $\mathcal{Y} = \{\{x\} : x \in X\}$ and assuming that the potential defector follows equilibrium behavior at stage-1), for any $\epsilon > 0$ there exist $N^5(\epsilon)$ and $\delta^5(\epsilon)$ such that if $|\sigma^N(x)/M^N - \mu_z(x)| < \delta^5(\epsilon)$ for all x and if $N > N^5(\epsilon)$, then $\| \beta^{\sigma^N}(x) - \beta^z(x) \| < \epsilon$ for all $x \in X$. The claim then follows from continuity of u and the law of large numbers, which implies that for any z and $x \in X$, $\lim_{N\to\infty} \sigma^N(x)/M^N = \mu_z(x)$ in probability conditional on z. ■

Given Claim 1 and Claim 3, the rest of the proof is the same as the proof of (i), except for letting $N(\kappa, \eta) = N^3(\kappa, \eta, \frac{\varepsilon}{3})$. □

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